

Permutation Patterns 2024

June 10-14, 2024

Moscow, ID, USA

LOCAL INFORMATION

Food and Coffee

Unfortunately, all campus food options are closed for the summer (some because buildings are undergoing renovation).

In addition to the options listed below, there is also fast food along the highway on the North edge of campus and fast food and other restaurants in the mall across from the Best Western. There are also some other restaurants in town that you could get to with a car.

Food and coffee options, listed in order going East along 6th street then North along Main street. We begin at the edge of campus, about a 10 minute walk from Renfrew Hall (and a little less from the dorm).

Patty's Mexican: Mexican food, counter service, all outdoor seating. Closed Tuesday and Wednesday

Stax: Sandwiches and soups. Open lunch only.

Taco Time: Mexican fast food.

Gambino's: Pizza and pasta.

Young's Alley: Small reasonably authentic Chinese restaurant: limited menu with only noodles, buns, and potstickers.

Moscow Alehouse: Beer and pub food.

Jimmy John's: the national sandwich chain.

Nectar: upscale American restaurant. Dinner only.

We are now at the corner of 6th and Main, about a 15 minute walk from Renfrew.

One World Cafe: coffee, open until 6. Also pastries, sandwiches, and beer. This is my usual coffee shop.

Maialina's: Pizza and pasta, fancier than Gambino's.

Mikey's Gyros: Counter service Greek food. Lunch only, closed Monday.

Humble Burger: Counter service burgers.

Breakfast Club: Eggs, omelettes, and sandwiches. Breakfast and lunch only.

La Casa Lopez: Mexican

Food Co-op (5th Street, one block East of Main): organic supermarket with a salad bar and hot bar.

Moscow Bagel and Deli: self-explanatory.

We are now at the corner of Main and 3rd, the center of town, about a 20 minute walk from Renfrew.

Cafe Artista: coffee.

Wooster's Pizzeria: pizza

Mad Greek: Greek food.

Tapped: good selection of beer and pub food.

Suchada's Thai (2nd Street, one block East of Main): Thai food.

Slice Taphouse (2nd Street, one block East of Main): Pizza and beer.

Malo (2nd Street, one block East of Main): Hawaiian. Closed Mondays.

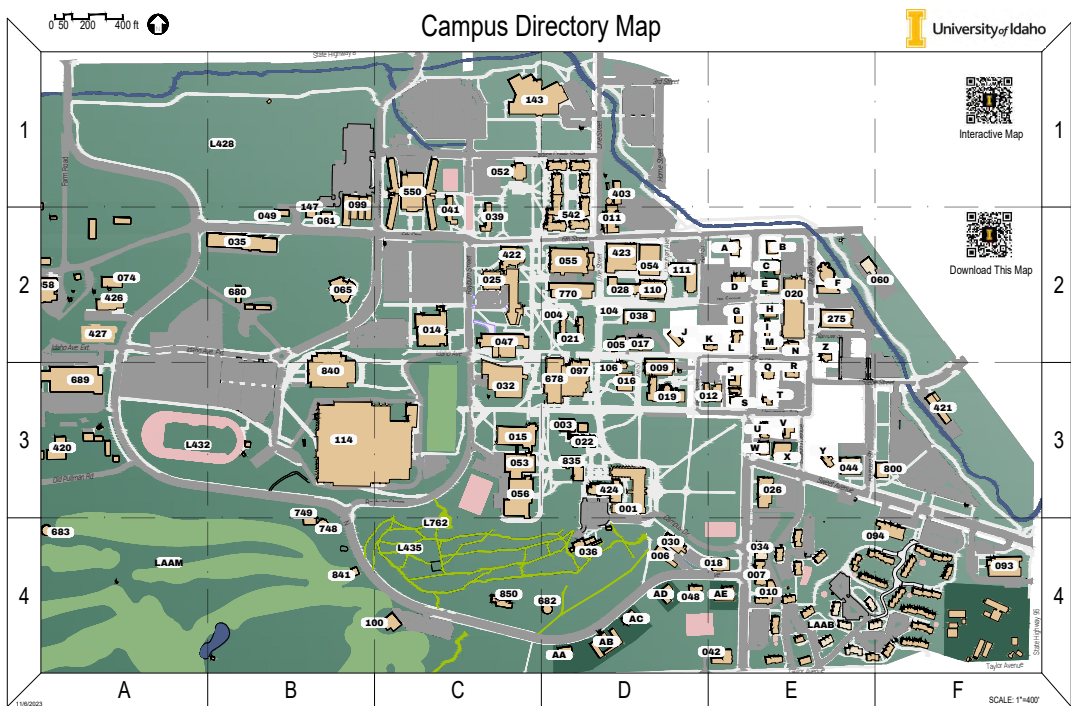
Bucer's Coffeehouse Pub: coffee, beer, sandwiches.

Lodgepole: the new upscale American place. Dinner only.

Parking

If you're staying in a dorm and have a car, parking information should have been e-mailed to you. Parking is free over summer EXCEPT in Gold lots and metered spaces. The closest free parking to Renfrew is the lot at the corner of 6th and Stadium; beware that only the third of the lot closest to Stadium is free (the divide is clearly marked).

Map



The dorm is among the buildings numbered 542 in the top middle. Renfrew (where the talks will be) is 047 not too far below that. IRIC (the social event Monday) is 770, roughly in between Renfrew and the dorm. The Best Western is just off the map at the upper left. Downtown is to the right.

CONFERENCE SCHEDULE

All events are in Renfrew 112 unless otherwise noted.

Monday

9:00–9:20: Registration

9:20–9:30: Welcome

9:30–10:00: Justin Troyka, *Asymptotic dichotomy of alternating sign matrices avoiding a permutation*

10:15–10:45: Noura Alshammari, *Asymptotic enumeration of monotone permutation grid classes*

11:00–11:30: Ben Jarvis, *$\mathcal{V}$ -antichains and applications to classes with bounded oscillations*

11:30–2:00: Lunch break

2:00–2:30: Michaela Polley, *$\top$ -avoiding rectangulations, inversion sequences, Dyck paths*

2:45–3:15: Kelvin Rivera-Lopez, *An algebraic construction of permuton-valued processes*

3:30–4:00: Anant Godbole, *The Expected Number of Distinct Non-Consecutive Patterns in a Random Permutation*

6:30–?? Estimathon! (in the IRIC Atrium)

Tuesday

9:30–10:00: Faqiruddin Azam, *Divisor labelling of staircase diagrams and fiber bundle structures on Schubert varieties*

10:15–11:15: Zachary Hamaker, *Permutation Patterns—the typical case*

11:20: conference photo

11:30–2:00: Lunch break

2:00–2:30: Bruce Sagan, *Difference d ascent sequences*

2:45–3:15: Giulio Cerbai, *Difference ascent sequences and d -Fishburn permutations*

3:30–4:00: Elizabeth Hartung, *Regular Word Pattern Avoidance for $(1, b)$ -Based Pell Numbers*

4:15–5:00: open problem session

Wednesday

9:30–10:00: Humaira Hameed, *On word-representability of (extended) Mycielski graphs*

10:15–10:45: Michal Opler, *The Hierarchy of Hereditary Sorting Operators*

11:00–11:30: Eric Marberg, *Permutation bialgebras from algebraic word relations*

2:00: Excursion

5:30: Banquet

Thursday

9:30–10:00: Erin McNicholas, *Cyclic sieving and homomesy on permutations: an analysis of maps and statistics in the FindStat database*

10:15–11:15: Lara Pudwell, *Gamifying Permutation Patterns*

11:30–2:00: lunch break

2:00–2:30: Amya Luo, *Pattern Avoidance in Nonnesting Permutations*

2:45–3:15: Reed Acton, *Arc Pattern Avoidance in Permutations*

3:30–4:00: Alex Burstein, *On shape-Wilf equivalence of certain sets of patterns*

Friday

9:30–10:00: Amrita Acharyya, *Type B Set partitions, an analogue of restricted growth functions*

10:15–10:45: Wang Xiao, *Linear independence of permutation as transformation for variables and transpose for tensors*

11:00–11:30: Saul Blanco, *Enumerating polynomial generalized permutation classes*

TYPE B SET PARTITIONS, AN ANALOGUE OF RESTRICTED GROWTH FUNCTIONS

Amrita Acharyya

In this work, we study type B set partitions for a given specific positive integer k defined over $\langle n \rangle = \{-n, -(n-1), \dots, -1, 0, 1, \dots, n-1, n\}$. We found a few generating functions of type B analogue for some of the set partition statistics defined by Wachs, White and Steingrimsson for partitions over positive integers $[n] = \{1, 2, \dots, n\}$, both for standard and ordered set partitions respectively. We extended the idea of restricted growth functions utilized by Wachs and White for set partitions over $[n]$, in the scenario of $\langle n \rangle$ and called the analogue as Signed Restricted Growth Function (SRGF).

Introduction

We initially describe the type-B analogue of ten set partition statistics over $\langle n \rangle$, which were defined by Steingrimsson over $[n]$ in case of standard and ordered both type of set partitions. We found the generating functions of the type B analogue of some of the set partition statistics defined over $\langle n \rangle$ which were defined by Steingrimsson over $[n]$, for any specified number of blocks k in terms of q -Stirling numbers of the second kind. There is a bijection between the set partitions of $[n]$ in standard form and restricted growth functions (RGF). Wachs and White defined four fundamental statistics on those RGFs. In this work, we defined an analogue of restricted growth function in case of type B set partitions of $\langle n \rangle$ and called it as Signed Restricted Growth Function. We found a similar kind of bijection between set partitions of $\langle n \rangle$ and SRGF. In [3] Steingrimsson defined ten set partition statistics over the set partitions of $[n]$. Four of these were already defined by Wachs and White as above in case of standard set partitions, and their treatment was in terms of restricted growth functions, a different way of representing partitions in standard form only. Another four statistics are mirror images of the aforementioned ones.

Steingrimsson's statistics over type -B partitions

Definition 1. [4] The type B q -Stirling numbers of the second kind are defined by the recurrence relation with

$S_B[n, k] = S_B[n-1, k-1] + [2k+1]S_B[n-1, k]$. The ordered version of $S_B[n, k]$ is $S_B^o[n, k] = [2k]!!S_B[n, k]$, where $[k]!! = [k][k-2][k-4] \dots$ ending at $[2]$ or $[1]$ depending on k is even or odd respectively.

Definition 2. (Sagan and Swanson)[4] A signed or type B partition is a partition of the set $\langle n \rangle = \{-n, \dots, -1, 0, 1, \dots, n\}$ of the form $\rho = S_0/S_1 \dots /S_k$, (We write $\rho \vdash_B \langle n \rangle$) satisfying $0 \in S_0$ and if $i \in S_0$, then $\bar{i} \in S_0$, where $\bar{i} = -i$ and for $i \geq 1$ we have $S_{2i} = -S_{2i-1}$, where $-S = \{-s | s \in S\}$.

Let $|S| = \{|s| : s \in S\}$, so that $|S_{2i}| = |S_{2i-1}|$ for $i \geq 1$. For all i we let $m_i = \min |S_i|$. Let $S_B(\langle n \rangle, k)$ denote the set of all type B partitions of $\langle n \rangle$ with $2k+1$ blocks in standard

form. We will always write signed partitions in standard form which means that $m_{2i} \in S_{2i}$ for all i , and $0 = m_0 < m_2 < m_4 < \dots < m_{2k}$. We define $M_i = \max|S_i|$, like m_i . For any $\rho \in S_B^0(\langle n \rangle, k)$

Definition 3.

- 1 $ros_B\pi = \#\{(s, S_j) | s \in S_i \text{ for some } i < j \text{ and } s \geq m_j\}$,
- 2 $rob_B\pi = \#\{(s, S_j) | s \in S_i \text{ for some } i < j \text{ and } s \leq m_j \text{ and } s > 0\}$,
- 3 $rcs_B\pi = \#\{(s, S_j) | s \in S_i \text{ for some } i < j \text{ and } s \geq M_j\}$,
- 4 $rcb_B\pi = \#\{(s, S_j) | s \in S_i \text{ for some } i < j \text{ and } s \leq M_j \text{ and } s > 0\}$,
- 5 $los_B\pi = \#\{(s, S_j) | s \in S_i \text{ for some } i > j > 0 \text{ and } s \geq m_j\}$,
- 6 $lob_B\pi = \#\{(s, S_j) | s \in S_i \text{ for some } i > j > 0 \text{ and } s \leq m_j \text{ and } s > 0\}$,
- 7 $lcs_B\pi = \#\{(s, S_j) | s \in S_i \text{ for some } i > j > 0 \text{ and } s \geq M_j\}$,
- 8 $lcb_B\pi = \#\{(s, S_j) | s \in S_i \text{ for some } i > j > 0 \text{ and } s \leq M_j \text{ and } s > 0\}$,
- 9 $rsb_B\pi = \#\{(s, S_j) | s \in S_i \text{ for some } i < j \text{ and } m_j \leq s \leq M_j\}$,
- 10 $lsb_B\pi = \#\{(s, S_j) | s \in S_i \text{ for some } i > j > 0 \text{ and } m_j \leq s \leq M_j\}$

Definition 4. For any $\pi \in S_B^0(\langle n \rangle, k)$, π^c is obtained by replacing any positive i by $n + 1 - i$, and $\bar{i}$ by $n + 1 - \bar{i}$ and keeping 0 the same.

Then it follows that $rcb\pi^c = ros\pi$ and that $rcs\pi^c = rob\pi$, as because each (s, S_j) contributing in $ros\pi$ gives $(n + 1 - s, S_j)$ contributing in $rcb\pi^c$ and conversely.

Lemma 5. The statistics ros_B and rcb_B are equidistributed over the ordered type B set partitions. The statistics rob_B and rcs_B are equidistributed over the ordered type B set partitions.

Theorem 6. $q^{k(k+1)}S_B[n, k] = \sum_{\rho \in S_B(\langle n \rangle, k)} q^{los_{B'}\rho}$ where $los_{B'}\rho = los_B\rho$ dropping the condition $j > 0$ in the definition of $los_B\rho$.

Theorem 7. We have $q^k S_B^0[n, k] = \sum_{\rho \in S_B^0(\langle n \rangle, k)} q^{los_{B'}\rho}$

Theorem 8. The generating function of $lob_{B'}$ over the standard type B partitions is given by $\sum_{\pi \in S_B[\langle n \rangle, k]} q^{lob_{B'}(\pi)} = S_B[n, k]q^k$. Additionally, the generating function of $lob_{B'}$ over the ordered type B partitions satisfies the following $\sum_{\pi \in S_B^0[\langle n \rangle, k]} q^{lob_{B'}(\pi)} = S_B[n, k]$. In particular, $\sum_{\pi \in S_B^0[\langle n \rangle, 1]} q^{lob_{B'}(\pi)} = S_B[n, 1]q + S_B[n, 1]$.

Signed Restricted Growth Functions[3]

Definition 9. (SRGF): A Signed Restricted Growth Function is a sequence of the form $w = a_0 a_1 a_1^* a_2 a_2^* \dots a_n a_n^*$ of length $2n + 1$, where

- i. $a_0 = 0$
- ii. If $a_i = j$, then $a_i^* = \bar{j}$ and conversely $\forall i, j \in \{1, 2 \dots n\}$.
- iii. $\forall i \geq 1, |a_{i+1}| \leq 1 + \text{Max}\{|a_0|, |a_1|, |a_2|, \dots |a_i|\}$
- iv. The pair $\bar{j}j$ appears before $j\bar{j}$ (if there is a $j\bar{j}$ in the sequence) for any $j \in \{1, 2 \dots n\}$

Calling the set of all such SRGF of length $2n+1$ as SR_n and $SR_{n,k}$ accordingly for SRGF of length $2n+1$ with maximal letter k , we can show that there is a bijection between the set of all type B partitions of $\langle n \rangle$ and SR_n , which preserves the one to one correspondence between $SR_{n,k}$ and $S_B(\langle n \rangle, k)$. The bijection is the following: Consider any type B partition π of $\langle n \rangle$. Associate with π , the word $w(\pi) = a_0 a_1 a_1^* a_2 a_2^* \dots a_n a_n^*$, where $a_0 = 0, a_i = a_i^* = 0$, iff $i, \bar{i} \in S_0$, for $i \in \{1, 2 \dots n\}$, $a_i = \bar{j}, a_i^* = j$ iff $i \in S_{2j}$ for $i, j \in \{1, 2 \dots n\}$ and $a_i = j, a_i^* = \bar{j}$ iff $i \in S_{2j-1}$ for $i, j \in \{1, 2 \dots n\}$

Question: Is there any way to find the generating functions of the type B-analogue of the ten statistics due to Steingrímsson for set partitions of $\langle n \rangle$ in standard form by using the correspondence with SRGF as in the approach in [2] for set partitions of $\langle n \rangle$ in standard form as in standard set partitions of $[n]$ in [5]? Can one do the same for ordered set partitions of $\langle n \rangle$ which is possibly without SRGF?

REFERENCES

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- [1] Michelle Wachs, Dennis White *p, q-Stirling Numbers and Set Partition Statistic* published by Journal of Combinatorial Theory (1991)
 - [2] Lindsey R. Campbell, Samantha Dahlberg, Robert Dorward, Jonathan Gerhard, Thomas Grubb, Carlin Purcell, Bruce E Sagan (2015) *Set partition patterns and statistics*
 - [3] Einar Steingrímsson (2019) *Statistics on ordered partitions of sets*
 - [4] Bruce E. Sagan And Joshua P. Swanson (2022) *q-STIRLING NUMBERS IN TYPE B*
 - [5] Bruce E. Sagan (1991) *A maj Statistic for Set Partitions*
 - [6] Ming-Jian Ding and Jiang Zeng. Some identities involving q-Stirling numbers of the second kind in type-B, The electronic journal of combinatorics, vol 31-issue 1

ARC PATTERN AVOIDANCE IN PERMUTATIONS

Reed Acton

Reykjavik University

This talk is based on joint work with Henning Ulfarsson

A *noncrossing arc diagram* or NCAD on n vertically arranged points is a collection of arcs where each arc moves monotone upwards from an endpoint p to a strictly higher endpoint q and passes to the left or to the right of each point between p and q [2]. Every pair of arcs α_1 and α_2 must satisfy the following conditions:

(C1) α_1 and α_2 cannot intersect in their interiors.

(C2) α_1 and α_2 cannot share an upper endpoint or a lower endpoint.

When considered up to combinatorial equivalence, the set $\mathfrak{E}_n$ of NCADs on n points is in bijection with the set S_n of permutations of size n . The bijection $\delta : S_n \rightarrow \mathfrak{E}_n$ maps the descending runs of a permutation $\omega \in S_n$ to the connected components of an NCAD $\mathcal{E} \in \mathfrak{E}_n$.

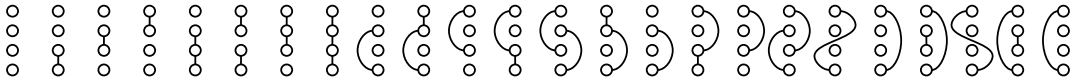


Figure 1: The 24 NCADs of $\mathfrak{E}_4$.

Since their introduction as a way to study lattice quotients of the weak order, NCADs have remained widely unexplored in other contexts. In this talk, we introduce the following definition of NCAD avoidance.

Definition 1. An NCAD A contains the NCAD B as a pattern if and only if there is a sequence $B = B_0, B_1, \dots, B_k = A$ where B_i can be obtained from B_{i-1} by removing either an arc α of B_{i-1} or removing a point p that is not an endpoint of any arc of B_{i-1} . We say A avoids B if there is no such sequence. We often refer to this property as *arc pattern avoidance*.

To understand NCAD avoidance, we turn to *vincular patterns*. A permutation $\pi \in S_n$ contains the vincular pattern (ω, X) if there is a instance $I = 1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that $\text{red}(\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_k}) = \omega$ and $i_{x+1} = i_x + 1$ for each $x \in X$. We prove the following result.

Theorem 2. Given a non-crossing arc diagram A , let U be the set of upper endpoints of arcs in A and let $\mathcal{A}$ be the set of non-crossing arc diagrams on the same number of points that contain A . Then, a non-crossing arc diagram B contains A if and only if $\delta^{-1}(B)$ contains the vincular pattern (ω, X) where the underlying permutation ω is $\omega = \delta^{-1}(C)$ for some $C \in \mathcal{A}$ and the set X of adjacencies is $X = \{k | \omega_k \in u\}$.

In Figure 2, we see how this theorem applies to the NCAD $A = \delta(3124)$.

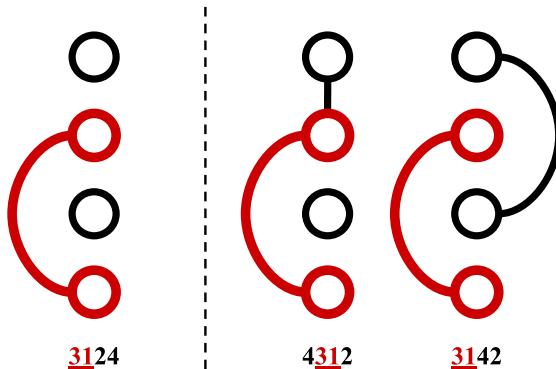


Figure 2: The set $\mathcal{A}$ for $\delta(3124)$ with corresponding vincular patterns below.

The set $\mathcal{A}$ contains every NCAD that can be obtained by adding arcs to A , and the vincular patterns have adjacencies that preserve the values and "shape" of the descents of 3124. Patterns such as $\underline{2}314$ have the same values as 3124 in their descent sets, but do not preserve the shape of this descent [1].

In addition to the introduction of arc pattern avoidance, we discuss symmetries and equivalence classes of arc pattern avoidance, and explore a number of fascinating enumerative results such as the following:

- The number of NCADS of size n whose set of avoiders can be described by a single vincular pattern is OEIS A307030.
- The arc pattern avoiders of $\delta(312) = \delta(231)$ are enumerated by the Catalan Numbers. The other size 3 NCADS do not share this property. A straightforward proof of this demonstrates that noncrossing arc diagrams can be thought of as a generalization of noncrossing partitions.
- The arc pattern avoiders of $\delta(24531) = \delta(53124)$ are enumerated by OEIS A328442: the number of inversion sequences of length n avoiding the consecutive pattern 210. We generalize this result and investigate deeper connections between noncrossing arc diagrams and inversion sequences.

We conclude with a discussion of other possible definitions of arc pattern avoidance and the utility of other bijections between S_n and $\mathfrak{E}_n$.

REFERENCES

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- [1] Einar Steingrímsson and Eric Babson. Generalized permutation patterns and a classification of the mahonian statistics. *Séminaire Lotharingien de Combinatoire* [electronic only], 44:B44b, 18 p., 2000.
 - [2] Nathan Reading. Noncrossing arc diagrams and canonical join representations. Preprint arXiv:1405.6904

ASYMPTOTIC ENUMERATION OF MONOTONE PERMUTATION GRID CLASSES

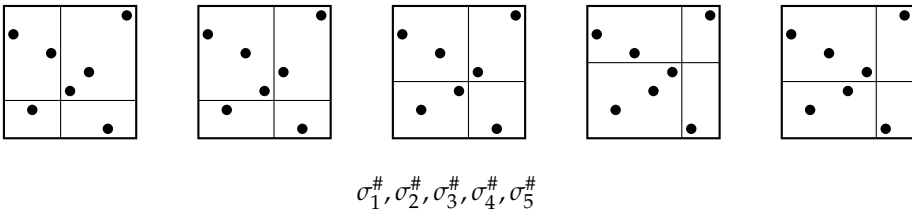
Noura Alshammari

University of Strathclyde

This talk is based on joint work with David Bevan

A monotone grid class $\text{Grid}(M)$ is a set of permutations whose shape is defined by a gridding matrix M with entries in $\{\begin{smallmatrix} \nearrow \\ \nwarrow \end{smallmatrix}, \begin{smallmatrix} \nwarrow \\ \nearrow \end{smallmatrix}, \square\}$. Each entry of the matrix corresponds to a cell c_{ij} in a gridding of a permutation. If the entry is $\begin{smallmatrix} \nearrow \\ \nwarrow \end{smallmatrix}$, then the points in the cell must increase. If it is $\begin{smallmatrix} \nwarrow \\ \nearrow \end{smallmatrix}$, they must decrease. If it is $\square$, then the cell must be empty.

We want to know the number of permutations of length n in $\text{Grid}(M)$, but the exact enumeration is hard, because permutations may have more than one gridding in the class. For example, the permutation $\sigma = 6253417$ in $\begin{smallmatrix} \nearrow \\ \nwarrow \end{smallmatrix}$ has 5 different gridgings:



The permutation $\sigma = 6253417$ belongs to the class, while the 5 griddings $\sigma_1^\#, \dots, \sigma_5^\#$ belong to the gridded class $\text{Grid}^\#(M)$. So, $\text{Grid}(M)$ is the set of permutations in the grid class, and the gridded class $\text{Grid}^\#(M)$ is the set of gridded permutations.

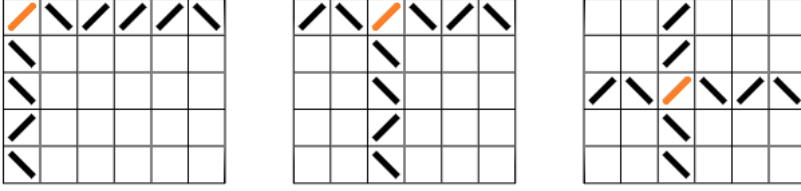
Finding the exact enumeration of grid classes is hard, so our goal is their asymptotic enumeration. Our strategy for the asymptotic enumeration of $\text{Grid}(M)$ consists of the following five steps:

1. Find the asymptotic enumeration of the gridded class: $|\text{Grid}_n^\#(M)| \sim \theta^\#(n)g^n$.
 - g is the exponential growth rate.
 - $\theta^\#$ is a subexponential term.
2. Find the proportion of points that occur in each cell in a typical large M -gridded permutation in $\text{Grid}^\#(M)$.
3. Find the structure that an M -gridded permutation $\sigma^\#$ must have so that σ has k distinct M -griddings.
4. Combine steps 2 and 3 to find

$$P_k = \lim_{n \rightarrow \infty} P[\sigma \text{ has } k \text{ distinct } M\text{-griddings} : \sigma^\# \in \text{Grid}_n^\#(M)]$$

5. Deduce the asymptotic enumeration $|\text{Grid}_n(M)| \sim c \cdot \theta^\#(n)g^n$, where $c = \sum_{k \geq 1} \frac{P_k}{k}$.

In particular, we consider L , T , and X -shaped classes with $r + 1$ rows and $c + 1$ columns with one corner cell. We say that a cell c is a **corner cell** or just **a corner** if there is another non-blank cell in the same row as c and there is another non-blank cell in the same column as c .



L , T , and X -shaped classes, showing the corner cell location

Step 1: Our approach is to determine the generating function for the gridded class, using a technique first described in [Bevan, 2015]. We then extract the asymptotic growth of the number of gridded permutations from the generating function using standard analytic combinatorics. In the case of the L , T , and X -shaped classes if M is connected with one corner and has dimensions $(r + 1) \times (c + 1)$, then

$$|\text{Grid}_n^\#(M)| \sim \theta^\# g^n, \text{ where } \theta^\# = \frac{r + c + q + 1}{2q} \text{ and } g = \frac{r + c + q + 1}{2},$$

with $q = \sqrt{(r + c + 1)^2 - 4cr}$

Step 2: For a connected one-corner class, the proportion of points in cells in a typical M -gridded permutation can be determined from Theorems 1 and 2.

Theorem 1. Suppose $\Gamma = (\gamma_{i,j})$ is a maximal M -distribution matrix. Then there exists a constant λ such that, for each nonzero entry $\gamma_{i,j}$ of Γ , we have

$$\frac{\gamma_{i,j}^2}{\rho_i \kappa_j} = \lambda,$$

where $\rho_i = \sum_j \gamma_{i,j}$ and $\kappa_j = \sum_i \gamma_{i,j}$ are the row and columns sums of Γ .

Theorem 2. If $\text{Grid}(M)$ is connected and $\Gamma_M = (\gamma_{i,j})$ is the unique maximal M -distribution matrix, then for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}[\max_{i,j} |\sigma_{(i,j)}^\# / n - \gamma_{i,j}| \leq \varepsilon : \sigma^\# \in \text{Grid}_n^\#(M)] = 1.$$

The unique maximal matrix of proportion of points in cells for the L -shaped class has the form

$$\begin{pmatrix} \alpha & \beta & \beta & \beta & \beta \\ \gamma & 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 & 0 \end{pmatrix}$$

proportions α, β , and γ

Then, by Theorems 1 and 2, we know that α, β , and γ are the unique positive solutions to the equations

$$\frac{\alpha^2}{(\alpha + r\gamma)(\alpha + c\beta)} = \frac{\beta}{(\alpha + c\beta)} = \frac{\gamma}{(\alpha + r\gamma)}, \text{ and } \alpha + c\beta + r\gamma = 1$$

Solving these implies

$$\alpha = \frac{1}{q}, \quad \beta = \frac{c - r + q - 1}{2cq}, \quad \gamma = \frac{r - c + q - 1}{2rq}$$

where

$$q = \sqrt{(r + c + 1)^2 - 4cr}.$$

Step 3: After that, for a typical large σ we find the structure of $\sigma^\#$ the ways in which permutations can be gridded must have so that σ has k distinct M griddings.

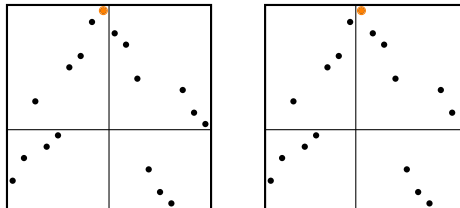
There are different types of corners. A peak is when the matrix contains a $\begin{smallmatrix} \diagup \\ \square \end{smallmatrix}$ followed by a $\begin{smallmatrix} \square \\ \diagdown \end{smallmatrix}$, or the opposite. For example, the class $\begin{smallmatrix} \diagup & \square \\ \square & \diagdown \end{smallmatrix}$ has one peak, and $\begin{smallmatrix} \square & \diagup \\ \diagdown & \square \end{smallmatrix}$ has two peaks. So, there are some classes with corner peaks which is the first type.

There is another type of classes with different corner cell which is called classes with long diagonal corner. The long diagonal corner in the class looks like $\begin{smallmatrix} \diagup & \diagup \\ \square & \square \end{smallmatrix}$, or $\begin{smallmatrix} \square & \square \\ \diagdown & \diagdown \end{smallmatrix}$.

For example, the class $\begin{smallmatrix} \diagup & \diagup \\ \square & \square \end{smallmatrix}$ has one long diagonal, and $\begin{smallmatrix} \square & \square \\ \diagdown & \diagdown \end{smallmatrix}$ has two long diagonals.

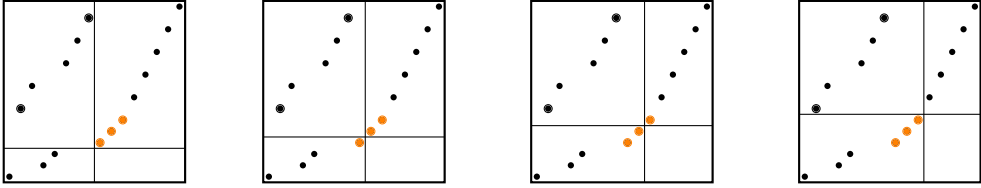
The third type is the Tee corner shape and it looks like $\begin{smallmatrix} \diagup & \diagup \\ \square & \diagdown \end{smallmatrix}$. For example, the class $\begin{smallmatrix} \square & \square \\ \square & \diagdown \end{smallmatrix}$ has one tee, and $\begin{smallmatrix} \square & \square \\ \diagdown & \square \end{smallmatrix}$ has two corner tees. So, the study included L, T , and X -shaped classes with a unique corner cell with the different types mentioned above.

There exist a dancing when a permutation has more than one gridding in the same class. In another meaning dancing is when a grid line can be moved from one side of a point(s) to the other side. This or these points are called the dancers. The peak point is the maximum point of the peak like $\begin{smallmatrix} \diagup & \square \\ \square & \diagdown \end{smallmatrix}$, or is the minimum of the peak like $\begin{smallmatrix} \square & \diagup \\ \diagdown & \square \end{smallmatrix}$. The peak point is a dancer if the grid line can be moved in either side of it. For example, the following permutation has two griddings in the class $\begin{smallmatrix} \diagup & \square \\ \square & \diagdown \end{smallmatrix}$ coming from the dancing of the (dancer) peak point.



Dancing of the peak point

Dancing in the diagonal corner may exist for more than one point. For example, the permutation in the class $\begin{smallmatrix} \diagup & \diagup \\ \square & \square \end{smallmatrix}$ has 3 dancers dancing like:



Diagonal dancing

Step 4: We then combine the two previous steps to find P_k where

$$P_k = \lim_{n \rightarrow \infty} P[\sigma \text{ has } k \text{ distinct } M\text{-griddings} : \sigma^\# \in \text{Grid}_n^\#(M)]$$

Step 5: Finally, we deduce the asymptotic enumeration $|\text{Grid}_n(M)| \sim c \cdot \theta^\#(n) g^n$, where $c = \sum_{k \geq 1} \frac{P_k}{k}$. Dividing each k distinct probability by k is to avoid the repetition of counting σ . This division guarantees that each permutation has been counted once and that is what we want.

DIVISOR LABELLING OF STAIRCASE DIAGRAMS AND FIBER BUNDLE STRUCTURES ON SCHUBERT VARIETIES

Faqrudin Azam

Lewis-Clark State University

Let $\text{Gr}(r, n)$ denote the Grassmannian of r -dimensional subspaces of $\mathbb{C}^n$. Each $\text{Gr}(r, n)$ contains a unique codimension-1 Schubert subvariety called the Schubert divisor of the Grassmannian. In this talk, we will discuss the correspondence between the set of permutations avoiding the patterns 3412, 52341, 52431 and 53241, and the set of Schubert varieties in the complete flag variety which are iterated fiber bundles of Grassmannians or Grassmannian Schubert divisors. Using this geometrical structure, we calculate the generating function that enumerates the permutations avoiding these patterns.

This talk is based on joint work with Daniel E. Skora

The *generalized symmetric group* is defined by the wreath product $S(m, n) := \mathbb{Z}_m \wr S_n$, where $\mathbb{Z}_m$ denotes the cyclic group of order m and S_n denotes the symmetric group of order $n!$. We extend the notion of permutation class for permutations, closed sets under containment, to generalized permutations. The enumeration and growth rate of permutation classes have long been studied. For instance, if $\mathcal{C}$ denotes a permutation class, then $|\mathcal{C} \cap S_n| < F_n$ for some n , if and only if $|\mathcal{C} \cap S_n|$ is eventually polynomial. This result is known as the *Fibonacci dichotomy*. We prove that the Fibonacci dichotomy also holds for generalized permutation classes and extend the known algorithm that produces the polynomial in the case of permutations to generalized permutations. We implement the algorithm and include applications to generalized permutation classes with fixed number of prefix reversals needed to sort them.

Introduction

In combinatorics, permutation classes are well-studied objects dating back to Knuth's *The Art of Computer Programming, Vol. 1* [5] where he studied stacks and dequeues. These permutation classes correspond to closed sets under containment, where σ is contained in π if one can find σ as a substring of π after standardization. A good survey of the field can be found in [6]. Our interest is related to questions regarding symbolic enumeration of permutation classes that are eventually polynomial. The so-called *Fibonacci dichotomy* serves as a characterization of these classes. In a few words, it states that if $\mathcal{C}$ denotes a permutation class and $\mathcal{C}_n = \mathcal{C} \cap S_n$, then $|\mathcal{C}_n|$ is eventually polynomial if and only if $|\mathcal{C}_N| \leq F_N$ for some N , where $\{F_n\}_{n \geq 0}$ denotes the Fibonacci numbers $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$. The Fibonacci dichotomy was first proved by Kaiser and Klazar [4] and later by Huczynska and Vatter [3]. Furthermore, Homberger and Vatter provided an algorithm that computes symbolically the polynomials for every permutation class that is eventually polynomial. Later, the authors studied enumeration of permutation classes for *signed permutations* [1]. These are permutations w of the set $\{-n, -(n-1), \dots, -1, 1, 2, \dots, n\}$ satisfying $w(-i) = -w(i)$ for $-n \leq i \leq n$.

In this paper, we extend the work in [3] for permutations and the work in [1] for signed permutations to generalized permutations. A key ingredient in the proof is the extension of *grid classes* from [3] to generalized permutations.

Our contribution

Our main contributions are the following

1. We extend the Fibonacci dichotomy to generalized permutation classes in $\mathbb{Z}_m \wr S_n$. In particular, we show that the characterization of polynomial permutation classes extends to generalized permutation classes that are eventually polynomial.
2. We extend the algorithm in [2] from S_n to $\mathbb{Z}_m \wr S_n$ and implement it in Python. The algorithm computes symbolically the polynomials arising from the enumeration of these generalized permutation classes.

A Python implementation can be found in the following link.

<https://github.com/danskora/SignedPermutationClasses>

Moreover, we show that the Fibonacci dichotomy holds for generalized permutation classes.

Theorem 1 (Generalized Fibonacci Dichotomy). *Let $\mathcal{C}$ be a generalized permutation class. Then following are equivalent.*

1. $|\mathcal{C} \cap S(m, n)|$ is eventually polynomial.
2. There is an integer N such that $|\mathcal{C}_N| < F_N$, where $\{F_n\}_{n \geq 0}$ denotes the Fibonacci numbers ($F_0 = 1, F_1 = 1, \dots$).

We then apply the algorithm to count the number of generalized permutations that require a certain number of *prefix reversals* to be sorted. In this context, we can write $\pi \in S(m, n)$ in one line notation as

$$\pi_1^{e_1} \pi_2^{e_2} \dots \pi_n^{e_n},$$

where $\pi_1 \pi_2 \dots \pi_n \in S_n$ and $e_1, \dots, e_n \in \mathbb{Z}_m$. Moreover, the prefix reversal $r_k : S(m, n) \rightarrow S(m, n)$ is defined by

$$r_k(\pi_1^{e_1} \pi_2^{e_2} \dots \pi_n^{e_n}) = \pi_k^{e'_k} \pi_{k-1}^{e'_{k-1}} \dots \pi_1^{e'_1} \pi_{k+1}^{e'_{k+1}} \dots \pi_n^{e_n},$$

where $e'_j = e_j + 1 \pmod m$. This type of sorting has connections to *parallel computing* and *genome rearrangements*.

Provided are several more enumerating polynomials obtained by the algorithm. Here, the polynomial $P_k^m(n)$ enumerates the class of generalized permutations that can be sorted in k or less prefix reversals.

$$\begin{aligned}
P_2^3(n) &= 1 + n + n^2, \\
P_3^3(n) &= 1 + n + n^3, \\
P_4^3(n) &= -8 + \frac{17}{2}n - \frac{1}{2}n^2 - n^3 + n^4, \\
P_5^3(n) &= 1 - \frac{75}{4}n + \frac{389}{24}n^2 - \frac{5}{4}n^3 - \frac{53}{24}n^4 + n^5, \\
P_3^4(n) &= 1 + 2n + n^3, \\
P_4^4(n) &= 1 + \frac{1}{2}n + \frac{5}{2}n^2 - n^3 + n^4, \\
P_5^4(n) &= \frac{10}{3}n - \frac{9}{4}n^2 + \frac{25}{6}n^3 - \frac{9}{4}n^4 + n^5, \\
P_4^5(n) &= 1 + n + 3n^2 - n^3 + n^4, \\
P_5^5(n) &= \frac{15}{4}n - \frac{67}{24}n^2 + \frac{21}{4}n^3 - \frac{53}{24}n^4 + n^5.
\end{aligned}$$

REFERENCES

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- [1] Saúl A. Blanco and Daniel E. Skora. An algorithm to enumerate grid signed permutation classes. In Alicia Dickenstein, Elias P. Tsigaridas, and Gabriela Jeronimo, editors, *Proceedings of the 2023 International Symposium on Symbolic and Algebraic Computation, ISSAC 2023, Tromsø, Norway, July 24-27, 2023*, pages 61–69. ACM, 2023.
 - [2] Cheyne Homberger and Vincent Vatter. On the effective and automatic enumeration of polynomial permutation classes. *Journal of Symbolic Computation*, 76:84–96, 2016.
 - [3] Sophie Huczynska and Vincent Vatter. Grid classes and the Fibonacci dichotomy for restricted permutations. *Electronic Journal of Combinatorics*, 13:Paper R54, 14 pp, 03 2006.
 - [4] Tomáš Kaiser and Martin Klazar. On growth rates of closed permutation classes. *Electron. J. Combin.*, 9(2):Paper R10, 20 pp, 2002-3. Permutation Patterns (Otago, 2003).
 - [5] Donald E. Knuth. *The Art of Computer Programming, Volume 1 (3rd ed.): fundamental algorithms*. Addison Wesley Longman Publishing Co., Inc., USA, 1997.
 - [6] Vincent Vatter. Permutation Classes. In Miklós Bóna, editor, *Handbook of Enumerative Combinatorics*, Discrete Mathematics and its Applications, chapter 12, pages 753–834. Chapman and Hall/CRC, 2015.

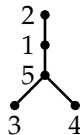
ON SHAPE-WILF EQUIVALENCE OF CERTAIN SETS OF PATTERNS

Alexander Burstein

Howard University

This talk is based on joint work with Tian Han, Sergey Kitaev, and Philip Zhang.

Gao and Kitaev [6] made a number of conjectures regarding relations between permutations avoiding partially ordered patterns (POPs) and other combinatorial objects (including permutations avoiding classical patterns). In particular, five out of six conjectures given in Table 5 in [6] were confirmed in Bean et al. [3] and Chen and Lin [5]. The remaining conjecture in [6, Table 5], related to the POP



and enumerated by the sequence in the OEIS [7], is, in terms of regular (non-POP) patterns, the Wilf-equivalence

$$\{12345, 12354\} \sim \{45123, 45213\}. \quad (1)$$

Bean et al. [3] used the Telescope software package [2] to generate the first 790 terms of the counting sequence of $\text{Av}(12345, 12354)$ and computed the first 50 terms of the counting sequence of $\text{Av}(45123, 45213)$ through different means. They also found six other classes, each avoiding two patterns of size 5, whose counting sequence matched in the first 100 terms. However, their attempt to solve conjecture (1) by Telescope led to a system of 23 equations with two catalytic variables, preventing the authors in [3] from solving it.

Our main result is proving the Wilf-equivalence (1) and hence settling the final conjecture in [6, Table 5]. To achieve this, we use the generalized versions of shape-Wilf-equivalence results of Backelin, West, and Xin [1], including shape-Wilf-equivalence of monotone patterns. We also derive general results on shape-Wilf-equivalence of certain classes of POPs and use their special case (also appearing in [4]) as an essential piece in proving (1). This special case is the fact that the sets of patterns $\{123, 213\}$ and $\{312, 321\}$ are shape-Wilf-equivalent on bottom-left-justified $(0, 1)$ -Ferrers boards. Our results let us show shape-Wilf-equivalence of large classes of sets of patterns such as

$$\begin{aligned} &\{4123, 4132, 4213, 4231, 4312, 4321\}, \\ &\{1423, 1432, 2413, 2431, 3412, 3421\}, \\ &\{1243, 1342, 2143, 2341, 3142, 3241\}, \\ &\{1234, 1324, 2134, 2314, 3124, 3214\}. \end{aligned}$$

REFERENCES

- [1] J. Backelin, J. West and G. Xin, Wilf-equivalence for singleton classes, *Adv. in Appl. Math.* **38** (2007), 133–148.
- [2] C. Bean, J. S. Eliasson, T. K. Magnusson, E. Nadeau, J. Pantone, H. Úlfarsson. Tilings: Combinatorial Exploration for permutation patterns. <https://github.com/PermutaTriangle/Tilings/>, June 2021. DOI: <https://doi.org/10.5281/zenodo.5810636>.
- [3] C. Bean, É. Nadeau, J. Pantone, H. Úlfarsson, Permutations avoiding bipartite partially ordered patterns have a regular insertion encoding, preprint, .
- [4] J. Bloom, S. Elizalde, Pattern avoidance in matchings and partitions, *Electron. J. Combin.* **20(2)** (2013), .
- [5] J. N. Chen, Z. Lin, A bijection for length-5 patterns in permutations. *J. Combin. Theory Ser. A* **202** (2024), Paper No. 105815, 34 pp.
- [6] Alice L. L. Gao and Sergey Kitaev, On partially ordered patterns of lengths 4 and 5 in permutations, *Electron. J. Combin.* **26(3)** (2019), .
- [7] OEIS, The Online Encyclopedia of Integer Sequences, .

This talk is based on joint work with Anders Claesson and Bruce Sagan

Dukes and Sagan [DS23] have recently introduced difference ascent sequences. Let $\alpha = a_1 \cdots a_n$ be a sequence of positive integers. Given a natural number $d \geq 0$, we call $i \in [n]$ a d -ascent if $i = 1$ or $i \geq 2$ and $a_i > a_{i-1} - d$. The d -ascent number is $\text{dasc}(\alpha) = |\{i \in [n] \mid i \text{ is a } d\text{-ascent of } \alpha\}|$. Then α is a d -ascent sequence if for all $i \in [n]$ we have $a_i \leq 1 + \text{asc}_d(a_1 \cdots a_{i-1})$. Let A_d denote the set of d -ascent sequences. For example, $d = 0$ gives classical ascent sequences [BMCDK10], while $d = 1$ gives weak ascent sequences [BCD23].

Let F_0 denote the set of permutations avoiding the bivincular pattern $\sigma_0 = (231, \{1\}, \{1\})$, called *Fishburn permutations*. The set F_0 has a tree-like structure in the sense that Fishburn permutations can be constructed recursively, starting from the empty permutation and repeatedly inserting a new maximum. By recording the label of the active site where the new maximum is inserted, we obtain a bijection $\Lambda_0 : F_0 \rightarrow A_0$. A non-recursive description of the inverse map of Λ_0 is obtained as $\Lambda_0^{-1} = \mathbf{t} \circ \text{hat}_0$, where $\mathbf{t}$ denotes the Burge transpose [3] and hat_0 denoted the hat map [BMCDK10]. The set $\hat{A}_0$ of *modified ascent sequences* is the bijective image of A_0 under hat_0 .

Here we introduce d -Fishburn permutations, F_d . They are characterized by avoidance of a marked bivincular pattern that is reminiscent of the original case. We extend the active site construction to any $d \geq 0$ by showing that the map Λ_d recording the active sites of d -Fishburn permutations is a bijection with dA . We introduce the d -hat map hat_d and obtain modified d -ascent sequences, $\hat{A}_d$, as the bijective image of dA under hat_d . By composing the d -hat with the Burge transpose, we obtain a non-recursive description of Λ_d^{-1} . Finally, we use the cluster method and a bijection with Dyck paths that do not contain ddu^{d+1} as a factor to enumerate the set $F_d(213)$ of d -Fishburn permutations avoiding 213.

d -Fishburn permutations

Let $d \geq 0$. Given $\pi = p_1 \cdots p_n \in \text{Sym}_n$, we define a step-by-step procedure to mark some entries of π . For $i = 1, \dots, n$, at the i th step i is d -marked if i appears to the left of $i - 1$ and there are at least d entries between i and $i - 1$ smaller than $i - 1$ that are not d -marked. We say that π contains the d -Fishburn pattern, denoted by $\mathbf{f}_d$, if π contains an occurrence $p_i p_j p_k$ of σ_0 where p_i is d -marked. Furthermore, we use the suggestive notation $F_d := \text{Sym}(\mathbf{f}_d)$ to denote the set of d -Fishburn permutations, defined as those that do not contain $\mathbf{f}_d$.

Theorem 1. *The set F_d has a tree-like structure. Furthermore, if $\pi = p_1 \cdots p_n \in F_d$, then the site before p_1 is always active and*

$$\text{the site after } p_i \text{ is active} \iff p_i \text{ is not } d\text{-marked.}$$

Theorem 2. The map Λ_d recording the active sites of permutations in F_d is a bijection from F_d to dA . Furthermore, the number of active sites of $\pi \in F_d$ is equal to $1 + \text{dasc}(\Lambda_d(\pi))$.

Modified d -ascent sequences

Let $d \geq 0$ and let $\alpha \in dA$. Then $\text{hat}_d(\alpha)$ is the sequence obtained by scanning the d -ascents of α from left to right; and, in each step, increasing by one every element strictly to the left of and weakly larger than the value of α at the current d -ascent. The set of *modified d -ascent sequences* is defined as $\hat{A}_d = \text{hat}_d(dA)$. A *Cayley permutation* is a sequence of positive integers that contains every number between 1 and its maximum value.

Proposition 3. Let $d \geq 0$ and $\alpha \in dA$. Then $\text{hat}_d(\alpha)$ is a Cayley permutation with $\max \text{hat}_d(\alpha) = \text{dasc}(\alpha)$. Furthermore, we have a bijection $\text{hat}_d : dA \rightarrow \hat{A}_d$.

Theorem 4. For every $d \geq 0$, the triangle

$$\begin{array}{ccc} A_d & \xrightarrow{\Lambda_d^{-1}} & F_d \\ & \searrow \text{hat}_d & \uparrow \tau \\ & & \hat{A}_d \end{array}$$

is commutative and all its arrows are size-preserving bijections.

Pattern avoidance in F_d

The introduction and characterization of the d -Fishburn permutations opens the door to pattern avoidance results parameterized by d . As an illustration, we enumerate $F_d(213)$.

Proposition 5. For any $d \geq 0$, we have a bijection between $F_d(213)$ and the set of Dyck paths that do not contain ddu^{d+1} as a factor.

We derive a generating function for the distribution of the number of ddu^{d+1} factors over Dyck paths. In the spirit of the cluster method [5, 6], we consider Dyck paths in which a subset of the ddu factors have been marked (with a q). Let

$$Q_d(q, x) = \sum_{\rho} (1 + q)^{\text{ddu}^{d+1}(\rho)} x^{|\rho|},$$

where the sum ranges over all Dyck paths, $|\rho|$ is the semilength of ρ , and $\text{ddu}^{d+1}(\rho)$ is short for the number of ddu^{d+1} factors in ρ . Then $Q_d(q, x)$ satisfies the functional equations:

$$\begin{aligned} Q_0(q, x) &= 1/(1 - xP_0(q, x)); \\ Q_d(q, x) &= \frac{1}{x \left(1 - \frac{xP_d(q, x)}{1 - xP_d(q, x)}\right)}, \quad d \geq 1, \end{aligned}$$

where $P_d(q, x)$ is defined by

$$\begin{aligned} P_0(q, x) &= 1 + xP_0(q, x)^2 + qx^2P_0(q, x)^2; \\ P_d(q, x) &= 1 + xP_d(q, x)^2 + qx^{d+1}P_d(q, x)^{d-1}, \quad d \geq 1. \end{aligned}$$

Finally, by Proposition 5, the generating function of $F_d(231)$ is

$$\sum_{\pi \in F_d(231)} x^{|\pi|} = Q_d(-1, x).$$

For a fixed small d one can derive an explicit expression for $Q_d(-1, x)$ by solving the corresponding system of functional equations. For instance,

$$Q_0(-1, x) = \frac{1-x}{1-2x} \quad \text{and} \quad Q_1(-1, x) = \frac{2(1-x)}{1-2x+x^2+\sqrt{1-4x+2x^2+x^4}}.$$

Since $|\text{Sym}_n(213)| = C_n$, the n th Catalan number, we find that the sequence of series $\{Q_d(-1, x)\}_{d \geq 0}$ converges to the generating function for the Catalan numbers:

$$\lim_{d \rightarrow \infty} Q_d(-1, x) = \frac{2}{1 + \sqrt{1-4x}}.$$

The coefficient of x^n in $Q_0(-1, x)$ is 2^{n-1} (for $n \geq 1$) and hence one might say that the coefficients in $Q_d(-1, x)$ “interpolate” between 2^{n-1} and C_n .

REFERENCES

-
- [1] Beáta Bényi, Anders Claesson, and Mark Dukes. Weak ascent sequences and related combinatorial structures. *European J. Combin.*, 108:Paper No. 103633, 19, 2023.
 - [2] Mireille Bousquet-Mélou, Anders Claesson, Mark Dukes, and Sergey Kitaev. (2+2)-free posets, ascent sequences and pattern avoiding permutations. *J. Combin. Theory Ser. A*, 117(7):884–909, 2010.
 - [3] Giulio Cerbai and Anders Claesson. Transport of patterns by Burge transpose. *European J. Combin.*, 108:Paper No. 103630, 25, 2023.
 - [4] Mark Dukes and Bruce Sagan. Difference ascent sequences, 2023. Preprint arXiv:2311.15370.
 - [5] Ian P. Goulden and D. M. Jackson. An inversion theorem for cluster decompositions of sequences with distinguished subsequences. *Journal of the London Mathematical Society*, 2(3):567–576, 1979.
 - [6] Chao-Jen Wang. *Applications of the Goulden-Jackson cluster method to counting Dyck paths by occurrences of subwords*. PhD thesis, Brandeis University, 2011.

THE EXPECTED NUMBER OF DISTINCT NON-CONSECUTIVE PATTERNS IN A RANDOM PERMUTATION

Anant Godbole

East Tennessee State University

This talk is based on joint work with Verónica Borrás-Serrano, Isabel Byrne, Carmen Jackson, Olivia LeBlanc, and Nathaniel Veimau

Let $\pi = \pi_n$ be a permutation on $[n]$, written in one-line notation. We say that π_n *consecutively contains* the pattern μ_k of length k if there are k consecutive indices $(m, m+1, \dots, m+k-1)$ such that $(\pi(m), \pi(m+1), \dots, \pi(m+k-1))$ are in the same relative order as $(\mu(1), \mu(2), \dots, \mu(k))$. We say that π_n *contains* a pattern $\mu = \mu_k$ of length k if there are k indices $n_1 < n_2 < \dots < n_k$ such that $(\pi(n_1), \pi(n_2), \dots, \pi(n_k))$ are in the same relative order as $(\mu(1), \mu(2), \dots, \mu(k))$. Let $\psi(\pi_n)$ and $\phi(\pi_n)$ be respectively the number of distinct *not-necessarily-consecutive* and *consecutive* patterns of all lengths $k; 1 \leq k \leq n$, contained in π_n . We focus on the case where π_n is a *uniformly chosen random permutation* on $[n]$, denote the random value of $\psi(\pi_n)$ or $\phi(\pi_n)$ (according to context) by $X = X_n$, and study the expected value $\mathbb{E}(X_n)$. In [3], it was shown that $\mathbb{E}(X_n)$ was close to its maximum value, i.e., $\frac{n^2}{2}(1 - o(1))$, in the consecutive case, and we wish to see what might be true in the non-consecutive case.

For context, first we summarize results on the extremal values of $\psi(\pi_n)$. The identity permutation reveals that

$$\min_{\pi_n \in S_n} \psi(\pi_n) = \min_{\pi_n \in S_n} \phi(\pi_n) = n + 1,$$

since the only embedded patterns are $\emptyset, 1, 12, \dots, (12 \dots n)$. On the other hand, motivated by a question posed by Herb Wilf at the inaugural Permutation Patterns meeting, held in Dunedin in 2003 (PP2003), several authors have studied the maximum value of $\psi(\pi_n)$. The trivial pigeonhole bound

$$\max_{\pi_n \in S_n} \psi(\pi_n) \leq \sum_{k=1}^n \min \left(\binom{n}{k}, k! \right) \sim 2^n, \quad (2)$$

was improved by several authors: [5], [4], and finally [7] in which Alison Miller showed that

$$2^n - O(n^2 2^{n-\sqrt{2n}}) \leq \max_{\pi_n \in S_n} \psi(\pi_n) \leq 2^n - \Theta(n 2^{n-\sqrt{2n}}). \quad (3)$$

The conjecture in [6] is that

$$\mathbb{E}(X_n) \sim 2^n,$$

i.e., that the expected value of X_n is again of the same order of magnitude as its maximum value. Though we are unable to prove (or disprove) this conjecture, this talk addresses several aspects of this question.

We start on a positive note by proving, using subadditivity arguments [2], [8], that

$$\lim \mathbb{E}(X_n)^{1/n} = c$$

exists and takes values in $[1, 2]$. The key step is to use Fekete’s lemma, in its improved Erdős-de Bruijn form [8]. We hope by June to have reduce the gap to the interval $[1.73, 2]$, with numerical evidence indicating that c may be higher than 1.836.

We then use basic counting methods are used to get close to proving the conjecture, and technical reasons are given as to what improvements might be needed to achieve our goal.

Finally, another possible route to resolving the conjecture is explored, and it is shown

$$\mathbb{E}(\Delta_n) \sim 2^{2n},$$

where $\Delta = \Delta_n$ is the number of pairs of non-isomorphic patterns. We indicate what work needs to be done to prove the conjecture from this result about pairs.

This research was supported by NSF Grants 1852171 and 2150434. REU students Borrás-Serrano, Byrne, and Veimau worked on the project in the Summer of 2021 and Spring of 2022, while REU students Jackson and Leblanc did the work on subadditivity in the Summer of 2023.

REFERENCES

-
- [1] Noga Alon and Joel Spencer, *The Probabilistic Method*, Fourth Edition, John Wiley, New York, 2016.
 - [2] Richard Arratia. On the Stanley-Wilf Conjecture for the Number of Permutations Avoiding a Given Pattern. *Electr. J. Combinatorics* 6, Paper N1, 1999.
 - [3] Austin Allen, Dylan Cruz, Veronica Dobbs, Egypt Downs, Evelyn Fokuoh, Anant Godbole, Sebastián Papanikolaou, Christopher Soto, and Lino Yoshikawa. The expected number of distinct consecutive patterns in a random permutation. *Pure Math. Appl. (PU. M. A.)* 30 (4), 1–10, 2022.
 - [4] Michael Albert, Micah Coleman, Ryan Flynn, Imre Leader. Permutations containing many patterns, *Ann. Comb.* 11 (3) 265–270, 2007.
 - [5] Micah Coleman. An answer to a question by Wilf on packing distinct patterns in a permutation. *Electron. J. Combin.*, 11 (1), 4 pp., 2004.
 - [6] Evelyn Fokuoh. The expected number of patterns in a randomly generated permutation on $[n] = \{1, 2, \dots, n\}$. M.S. Thesis, East Tennessee State University, 2018, available at <https://dc.etsu.edu/cgi/viewcontent.cgi?article=4916&context=etd>.
 - [7] Alison Miller. Asymptotic bounds for permutations containing many different patterns. *J. Comb. Theory, Series A*, 116 (1), 92–108, 2009.
 - [8] J. Michael Steele *Probability Theory and Combinatorial Optimization*, CBMS-NSF Regional Conference Series in Applied Mathematics 69, SIAM, Philadelphia, PA., 1997.

PERMUTATION PATTERNS—THE TYPICAL CASE

Zachary Hamaker

University of Florida

In permutation patterns, most research is concerned with properties of permutations avoiding a given pattern. There is also a rich literature on the opposite extreme: packing as many instances of a pattern into a permutation as is possible. In this talk, I will survey the middle ground of pattern counts in the typical case. Many variations on this theme will be explored: notions of a pattern, properties to investigate and interpretations of ‘typical’. I will give special attention to recent work relating patterns and cycle type, including some original work joint with Brendon Rhoades.

There is a long line of research dedicated to representing graphs using patterns in words. In particular, the theory of word-representable graphs is relevant to algebra, graph theory, computer science, combinatorics on words, and scheduling [4], and these graphs generalize several fundamental classes of graphs such as *circle graphs*, *3-colorable graphs* and *comparability graphs*. Also, the notion of a k -11-representable graph was introduced by Jeff Remmel in 2017 and studied by Cheon et al. [1] in 2019 as a natural extension of word-representable graphs (the same as 0-11-representable graphs). A graph is k -11-representable if it can be represented by a word w such that for any edge (resp., non-edge) xy in G the subsequence of w formed by x and y contains at most k (resp., at least $k + 1$) occurrences of the consecutive pattern 11. A remarkable result of Cheon et al. [1] shows that any graph is 2-11-representable, while it is unknown whether every graph is 1-11-representable.

The Mycielski graph of an undirected graph is a larger graph, constructed in a certain way, that maintains the property of being triangle-free but enlarges the chromatic number. These graphs are important as they allow to prove the existence of triangle-free graphs with arbitrarily large chromatic number. An extended Mycielski graph is a certain natural extension of Mycielski graphs.

In this presentation, based on the work in [3], I will show how to characterize completely word-representable extended Mycielski graphs and comparability Mycielski graphs. A complete characterization of word-representable Mycielski graphs is also conjectured. These studies are a far-reaching extension of the result of Kitaev and Pyatkin [5] on non-word-representability of the Mycielski graph $\mu(C_5)$ of the cycle graph C_5 . Using a recent result of Kitaev and Sun [6], it is possible to shorten the length of the original proof of non-word-representability of $\mu(C_5)$ from 2 pages to a few lines.

Finally, a recent result of Futorny et al. [2] demonstrates that a non-word-representable graph $\mu(C_{2n+1})$ is actually 1-11-representable, a topic I will also discuss in my presentation.

REFERENCES

- [1] G.-S. Cheon, J. Kim, M. Kim, S. Kitaev and A. Pyatkin. On k -11-representable graphs, *J. Combin.* **10** (2019) 3, 491–513.
- [2] M. Futorny, S. Kitaev and A. Pyatkin. New tools to study 1-11-representation of graphs, submitted for publication, 2024.
- [3] H. Hameed. On semi-transitivity of (extended) Mycielski graphs, submitted for publication, 2023.
- [4] S. Kitaev and V. Lozin. Words and Graphs, *Springer*, 2015.

- [5] S. Kitaev and A. Pyatkin. On semi-transitive orientability of triangle-free graphs. *Discussiones Mathematicae Graph Theory* **43** (2023), ID: 4621, page 533.
- [6] S. Kitaev and H. Sun. Human-verifiable proofs in the theory of word-representable graphs. *RAIRO-Theor. Inf. Appl.* **58** (2024).

This talk is based on joint work with Aaron Williams

The well-known *Pell numbers* follow the recurrence $P(n) = 2 \cdot P(n-1) + P(n-2)$ with base cases $P(1) = 1$ and $P(2) = 2$. We consider the $(1, b)$ -based *Pell numbers* $P_b(n)$ in which the base cases are $P_b(1) = 1$ and $P_b(2) = b$. We show that $P_b(n) = |\text{Av}_n^{b-1}(132, 221)|$ for all $b \geq 3$. In other words, the $(1, b)$ -based Pell numbers enumerate the $(b-1)$ -regular words over $[n]$ that avoid the pattern 132 and the multi-pattern 221. We also discuss two simpler enumerations results: $|\text{Av}_n^r(132, 211)|$ and $|\text{Av}_n^r(123, 211)|$.

Regular Word Pattern Avoidance for Parameterized Sequences

An r -regular word over $[n] = \{1, 2, \dots, n\}$ is a word of length rn in which each symbol in $[n]$ appears r times. We let S_n^r be the set of all such words. For example, $S_2^2 = \{1122, 1212, 1221, 2112, 2121, 2211\}$ and in general $|S_n^r| = (rn)! / (r!)^n$. Note that permutations are the special case of $r = 1$ (i.e., $S_n = S_n^1$) and various other terms have been used when $r > 1$ (e.g., *uniform words* and *fixed-frequency multiset permutations*).

We use Av_n^r for the subset of S_n^r that avoids a list of patterns, with the assumption that readers are familiar with background concepts and notation from [1]. For example, k -Stirling permutations are typically defined as k -regular words in which any symbols between two copies of $i \in [n]$ are larger than i ; equivalently, they are $\text{Av}_n^k(212)$.

Regular words can contain patterns with repeated symbols. Let M_n^m be the set of *multi-patterns* of length n containing each symbol in $[m]$ at least once. For example, $M_3^2 = \{112, 121, 122, 211, 212, 221\}$ contains the multi-patterns with 2 distinct symbols and length 3. Standard patterns arise when $m = n$ (i.e., $M_n^n = S_n$).

While the avoidance of short patterns in permutations has been studied exhaustively, the same is not true of regular words. An exception is Kuba and Panholzer's study of k -Stirling permutations avoiding a subset of S_3 [6]. One of their results is that Fibonacci- k numbers count $\text{Av}_n^k(123, 132, 213, 121)$. More broadly, regular words hold the promise of being enumerated by *parameterized sequences* rather than single sequences. Here we focus on parameterized sequences obtained from avoiding one length 3 pattern $\alpha \in S_3$ and one length 3 multi-pattern $\beta \in M_3^2$, with recent results summarized below.

- k -Catalan numbers count $\text{Av}_n^{k-1}(\alpha, \beta)$ for *agreeable pairs*: $(123, 112)$ [8], $(231, 121)$ [6], and $(231, 221)$ [2]. The three cases give the MacMahon-Knuth theorem for $k = 2$ as $\text{Av}_n(123) = \text{Av}_n^1(123, 112)$ [7] and $\text{Av}_n(231) = \text{Av}_n^1(231, 121) = \text{Av}_n^1(231, 221)$ [5].
- k -Fibonacci numbers enumerate $\text{Av}_n^k(\alpha, \beta)$ for $\alpha = 213$ and $\beta = 122$ [3].

We resolve three more cases of this ilk. It's worth noting that Theorem 1 is focused on Pell numbers that have a *parameterized base case*. This differs from the k -Fibonacci / Fibonacci- k results that have *parameterized inductive formulae* [3].

(1, b)-Based Pell Numbers

The $(1, b)$ -based Pell numbers are defined inductively for $n \geq 1$ as

$$P_b(n) = 2 \cdot P_b(n-1) + P_b(n-2) \text{ with } P_b(1) = 1 \text{ and } P_b(2) = b. \quad (4)$$

Pell numbers $P(n)$ use $b = 2$ (i.e., $P(n) = P_2(n)$), and the Pell sequence (OEIS A000129) includes $P(0) = 0$. Table 1 illustrates the parameterized sequence and our first result.

Theorem 1. $P_b(n) = |\text{Av}_n^{b-1}(132, 221)|$ for all $b \geq 3$ and $n \geq 1$. That is, the $(1, b)$ -based Pell numbers enumerate $(b-1)$ -regular words over $[n]$ avoiding 132 and 221 for $b \geq 3$ and $n \geq 1$.

Proof. Let $n \geq 3$. If $\alpha \in \text{Av}_n^{b-1}(132, 221)$ and $x, y \in [n]$ with $x < y$, then all copies of x precede y^{b-2} ; otherwise, α contains 221. Thus, all $x < n-1$ precede $(n-1)^{b-2}n^{b-2}$ in α . Consider the $(b-1)$ th n and $n-1$ in α . For $x < n-1$, the substring $xn(n-1)$ is the forbidden 132, so only $n-1$ can precede n . Thus, α is one of the following: $\alpha = (n-1)n\gamma(n-1)^{b-2}n^{b-2}$ where $\gamma \in \text{Av}_{n-2}^{b-1}(132, 221)$, or $\alpha = \beta n^{b-1}$ or $n\beta n^{b-2}$ where $\beta \in \text{Av}_{n-1}^{b-1}(132, 221)$. Each $\gamma \in \text{Av}_{n-2}^{b-1}(132, 221)$ results in one word in $\text{Av}_n^{b-1}(132, 221)$; each $\beta \in \text{Av}_{n-1}^{b-1}(132, 221)$ results in two words in $\text{Av}_n^{b-1}(132, 221)$. Together with the base cases $|\text{Av}_1^{b-1}(132, 221)| = 1$ and $|\text{Av}_2^{b-1}(132, 221)| = b$, this proves the result. $\square$

Sequence	b	Numbers	OEIS	r	Pattern Avoidance
Catalan	$\times$	1, 1, 2, 5, 14, 42, 132, 429, ...	A000108	1	$\text{Av}_n(132)$
Pell	2	1, 2, 5, 12, 29, 70, 169, 408, ...	A000129	$\times$	132-avoiding two-stack sortable
(1, 3)-based Pell	3	1, 3, 7, 17, 41, 99, 239, 577, ...	A001333	2	$\text{Av}_n^2(132, 221)$
(1, 4)-based Pell	4	1, 4, 9, 22, 53, 128, 309, 746, ...	A048654	3	$\text{Av}_n^3(132, 221)$
(1, 5)-based Pell	5	1, 5, 11, 27, 65, 157, 379, 915, ...	A048655	4	$\text{Av}_n^4(132, 221)$
(1, 6)-based Pell	6	1, 6, 13, 32, 77, 186, 449, 1084, ...	A048693	5	$\text{Av}_n^5(132, 221)$
(1, 7)-based Pell	7	1, 7, 15, 37, 89, 215, 519, 1253, ...	A048694	6	$\text{Av}_n^6(132, 221)$
(1, 8)-based Pell	8	1, 8, 17, 42, 101, 244, 589, 1422, ...	A048695	7	$\text{Av}_n^7(132, 221)$

Table 1: Pell and $(1, b)$ -based Pell numbers for $3 \leq b \leq 9$. Theorem 1 proves that the latter count r -regular words avoiding 132 and 221 for $b = r + 1 \geq 3$. When $b = 2$ the words are counted by Catalan numbers while Pell numbers count 132-avoiding two-stack sortable permutations [4].

Two Additional Parameterized Sequences

We complete this work with two smaller results on avoiding an $\alpha \in S_3$ and a $\beta \in M_3^2$. For a summary of other such results (and conjectures) see Table 5 in [3].

Theorem 2. $|\text{Av}_n^r(132, 211)| = r(n-1) + 1$ for all $r \geq 2$ and $n \geq 1$.

Sketch. Deleting the rightmost 1 in $w \in \text{Av}_n^r(132, 211)$ gives $w' = 1^{r-1}2^r3^r \cdots n^r$. Otherwise, w' has a leftmost decrease xy . If $y = 1$, then w contains 211 by the deleted 1. If $y > 1$, then w contains 132 by an earlier 1, or 211 by later 1s. Finally, the $r(n-1) + 1$ distinct strings in $\text{Av}_n^r(132, 211)$ come from inserting a rightmost 1 in $1^{r-1}2^r3^r \cdots n^r$. $\square$

Our last remark follows from Figure 7 and $\text{Av}_n^r(123, 211) \subseteq \text{Av}_n^r(123, 21) = \emptyset$ for $n \geq 3$.

Remark 3. $\{|\text{Av}_n^r(123, 211)|\}_{n \geq 1} = 1, r + 1, 0, 0, 0, \dots$ for all $r \geq 2$.

$Av_1^3(132, 221)$	$Av_4^3(132, 221)$	$Av_1^3(132, 211)$	$Av_2^3(132, 211)$
111	111222333444	111	111222
total: $P_4(1) = 1$	112122333444	total: $r(n-1) + 1 = 1$	112122
	121122333444		112212
	211122333444	$Av_4^3(132, 211)$	112221
$Av_2^3(132, 221)$	231112233444	111222333444	total: $r(n-1) + 1 = 4$
111222	311122333444	112122333444	
112122	311212233444	112212333444	$Av_3^3(132, 211)$
121122	312112233444	112221333444	111222333
211122	321112233444	112223133444	112122333
total: $P_4(2) = 4$	341112223344	112223313444	112212333
	341121223344	112223331444	112221333
	341211223344	112223334144	112223133
$Av_3^3(132, 221)$	342111223344	112223334414	112223313
111222333	411122233344	112223334441	112223331
112122333	411212233344	total: $r(n-1) + 1 = 10$	total: $r(n-1) + 1 = 7$
121122333	412112233344		
211122333	421112233344		
231112233	423111223344		
311122233	431112223344		
311212233	431121223344		
312112233	431211223344		
321112233	432111223344		
total: $P_4(3) = 9$	total: $P_4(4) = 22$		

$Av_1^3(123, 211)$	$Av_2^3(123, 211)$	$Av_3^3(123, 211)$
111	111222	(none)
total: 1	112122	total: 0
	112212	$Av_4^3(123, 211)$
	112221	(none)
	total: $r + 1 = 4$	total: 0

Figure 7: Illustrating Theorem 1 (left), Theorem 2 (top-right), and Remark 3 (bottom-right) for $r = 3$ and $1 \leq n \leq 4$. Note: $r = 3$ corresponds to the $(1, 4)$ -based Pell numbers in Theorem 1.

REFERENCES

- [1] David Bevan. Permutation patterns: basic definitions and notation. arXiv preprint arXiv:1506.06673, 2015.
- [2] Colin Defant and Noah Kravitz. Stack-sorting for words. *Australasian Journal of Combinatorics*, 77(1):51–68, 2020.
- [3] Emily Downing, Elizabeth Hartung, and Aaron Williams. Pattern avoidance for Fibonacci sequences using k-regular words. arXiv preprint arXiv:2312.16052, 2023.
- [4] Eric S Egge and Toufik Mansour. 132-avoiding two-stack sortable permutations, Fibonacci numbers, and Pell numbers. *Discrete applied mathematics*, 143:72–83, 2004.
- [5] Donald E Knuth. The Art of Computer Programming, Vol 1: Fundamental Algorithms. *Algorithms*. Reading, MA: Addison-Wesley, 1968.
- [6] Markus Kuba and Alois Panholzer. Enumeration formulae for pattern restricted Stirling permutations. *Discrete Mathematics*, 312(21):3179–3194, 2012.
- [7] Percy Alexander MacMahon. *Combinatory analysis, volumes I and II*, volume 1. American Mathematical Soc., 2001.

- [8] Aaron Williams. Pattern avoidance for k -Catalan sequences. In *Proceedings of the 21st International Conference on Permutation Patterns*, pages 147–149, 2023.

This talk is based on joint work with Robert Brignall

The existence of the infinite antichain of oscillations at growth rate $\kappa \approx 2.20557$ marks an important phase transition in the set of possible growth rates of permutation classes. This is the smallest infinite antichain, and Vatter [1] and Bevan [2] have used it to construct permutation classes of every growth rate greater than $\lambda_A \approx 2.35698$. On some level, this construction feels like a cheat: we suspect that there is some further structure in the set of possible growth rates greater than λ_A that is being obscured by the antichain of oscillations and the myriad classes it can be used to construct. This has motivated the study of the universe of permutation classes with bounded oscillations, and, in particular, the infinite antichains which avoid long oscillations. We begin with a discussion of the antichain $\mathcal{V}$ - believed to be in some sense the second-smallest infinite antichain - at growth rate $\nu \approx 3.069$ and its use in Bevan-like constructions to construct intervals of growth rates which can be achieved by permutation classes with bounded oscillations. We move on to an attempt to generalise $\mathcal{V}$ using pin sequences (a construction first introduced by Brignall, Huczynska and Vatter [3] as a means of studying simple permutations): this results in a method for turning binary sequences into permutation classes containing ‘genuinely distinct’ infinite antichains. We shall discuss a remarkable feature of these so-called $\mathcal{V}$ -classes: their growth rates depend only on the *recurrent complexity* of the defining binary sequence (the sequence counting the number of factors that recur infinitely often). This results in extraordinarily large collections of distinct infinite antichains occurring at the same growth rates, and will allow us to identify an interesting new point of phase transition in the set of growth rates of permutation classes, at a constant $\mathcal{L} \approx 3.28$. We shall show that this constant $\mathcal{L}$ is an accumulation point of genuinely distinct infinite antichains from both above and below, as well as demonstrating the appearance at $\mathcal{L}$ of uncountably many permutation classes with distinct enumeration sequences. Amongst other consequences, the existence of uncountably many infinite antichains by growth rate $\mathcal{L}$ places an upper limit on the viability of banning small antichains as a way of investigating the deeper structure of the set of growth rates of permutation classes: there are simply too many antichains by this point. We conclude with a brief discussion of possible further directions in the ongoing classification of infinite antichains.

REFERENCES

- [1] Vincent Vatter. Permutation classes of every growth rate above 2.48188. *Mathematika*, 56(1):182–192, 2010.
- [2] David Bevan. Intervals of permutation class growth rates. *Combinatorica*, 38(2):279–303, 2018

- [3] Robert Brignall, Sophie Huczynska, and Vincent Vatter. Decomposing simple permutations, with enumerative consequences. *Combinatorica* , 28:385–400, 2008.
- [4] Frédérique Bassino, Mathilde Bouvel, and Dominique Rossin. Enumeration of pin-permutations. *Electron. J. Combin.* , 18(1):Paper 57, 39, 2011.
- [5] M.H. Albert, A. Atminas, and R. Brignall. Characterising inflations of monotone grid classes of permutations. *J. Combin. Theory Ser. A*, 154:444–463, 2018.

PATTERN AVOIDANCE IN NONNESTING PERMUTATIONS

Amya Luo

Dartmouth College

This talk is based on joint work with Sergi Elizalde

Nonnesting permutations are permutations of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ that avoid subsequences of the form $abba$ for any $a \neq b$. These permutations have recently been studied in connection to noncrossing (also called quasi-Stirling) permutations, which are those that avoid subsequences of the form $abab$, and in turn generalize the well-known Stirling permutations. Inspired by the work by Archer et al. on pattern avoidance in noncrossing permutations, we consider the analogous problem in the nonnesting case. Specifically, we enumerate nonnesting permutations that avoid each set of two or more patterns of length 3, obtaining closed formulas and generating functions in each case. Our proofs use recurrences and bijections.

Definitions

Let $\mathcal{S}_n$ be the set of permutations of $[n] := \{1, 2, \dots, n\}$. Denote by $[n] \sqcup [n] = \{1, 1, 2, 2, \dots, n, n\}$ the multiset consisting of two copies of each integer between 1 and n .

Given two words $\pi = \pi_1 \pi_2 \dots \pi_m$ and $\sigma = \sigma_1 \sigma_2 \dots \sigma_k$ over the positive integers $\mathbb{N}$, we say that π *contains* the pattern σ if there exist indices $1 \leq i_1 < i_2 < \dots < i_k \leq m$ such that the subsequence $\pi_{i_1} \pi_{i_2} \dots \pi_{i_k}$ is in the same relative order as σ , that is, $\pi_{i_r} < \pi_{i_s}$ if and only if $\sigma_r < \sigma_s$, and $\pi_{i_r} = \pi_{i_s}$ if and only if $\sigma_r = \sigma_s$ for all $r, s \in [k]$. If π does not contain σ , we say that π *avoids* the pattern σ .

In the 70s, Gessel and Stanley [2] introduced *Stirling permutation*, which is a permutation π of $[n] \sqcup [n]$ that avoids the pattern 212; equivalently, there do not exist indices $1 \leq i_1 < i_2 < i_3 \leq 2n$ such that $\pi_{i_2} < \pi_{i_1} = \pi_{i_3}$. To generalize Stirling permutations, Archer et al. [1] introduced *quasi-Stirling permutation*, which is a permutation π of $[n] \sqcup [n]$ that avoids the patterns 1212 and 2121; equivalently, there do not exist indices $1 \leq i_1 < i_2 < i_3 < i_4 \leq 2n$ such that $\pi_{i_1} = \pi_{i_3}$ and $\pi_{i_2} = \pi_{i_4}$.

One can view permutations π of $[n] \sqcup [n]$ as labeled matchings of $[2n]$, by placing an arc with label ℓ between i and j if $\pi_i = \pi_j = \ell$. With this interpretation, a permutation of $[n] \sqcup [n]$ is quasi-Stirling if and only if the corresponding matching is noncrossing.

It is then natural to consider permutations of $[n] \sqcup [n]$ whose corresponding matching is nonnesting. We denote by $\mathcal{C}_n$ the set of nonnesting permutations of $[n] \sqcup [n]$. It is well known that both noncrossing and nonnesting matchings of $[2n]$ are counted by the n th Catalan number $\text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}$. Since there are $n!$ ways to assign the labels to the arcs, it follows [3] that both the number noncrossing and the number of nonnesting permutations of $[n] \sqcup [n]$ are given by

$$|\mathcal{C}_n| = n! \text{Cat}_n = \frac{(2n)!}{(n+1)!}.$$

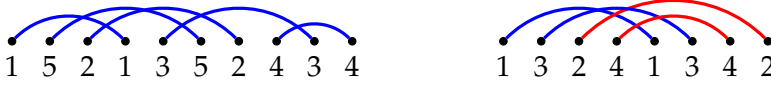


Figure 8: The matchings corresponding to $1523415234 \in \mathcal{C}_n$ and $13241342 \notin \mathcal{C}_n$, respectively.

In [1], Archer et al. consider quasi-Stirling permutations that avoid any set of at least two elements from $\mathcal{S}_3$. The goal of this paper is to extend the results from [1] to the nonnesting case, as well as enumerating nonnesting permutations that avoid a few other patterns.

Given a set Λ of finite words over $\mathbb{N}$, let $\mathcal{C}_n(\Lambda)$ denote the set of permutations in $\mathcal{C}_n$ that avoid all the patterns in Λ , and let its cardinality be $s_n(\Lambda) = |\mathcal{C}_n(\Lambda)|$.

For any word $\alpha = \alpha_1\alpha_2 \dots \alpha_k$ over $\mathbb{N}$, define its reversal by $\alpha^r = \alpha_k \dots \alpha_2\alpha_1$. If n is the largest entry of α , define its complement α^c to be the word whose i th entry is $n + 1 - \alpha_i$ for all $i \in [k]$. Clearly, π avoids σ if and only if π^r avoids σ^r , and also if and only if π^c avoids σ^c . In particular, the set $\mathcal{C}_n$ is closed under the operations of reversal and complementation.

Main Results

We enumerated nonnesting permutations avoiding at least two patterns of length 3. Applying reversal and complementation, the enumeration of $\mathcal{C}_n(\Lambda)$ for all $\Lambda \subseteq \mathcal{S}_3$ with $|\Lambda| \geq 2$ can be reduced to the sets listed in Table 2.

We also studied nonnesting permutations that avoid other patterns. The results and conjectures are summarized in Table 3 and Table 4, respectively.

$\Lambda \subseteq \mathcal{S}_3$	Formula for $s_n(\Lambda)$
$\{123, 321\}$	0, for $n \geq 5$
$\{123, 231\}$	$\frac{(n-1)(n+6)}{2}$, for $n \geq 2$
$\{132, 213\}$	F_n^2
$\{132, 231\}$	2^n , for $n \geq 2$
$\{132, 312\}$	$4 \cdot 3^{n-2}$, for $n \geq 2$
$\{312, 321\}$	
$\{123, 132, 213\}$	o.g.f.: $A(x) = \frac{1-x}{1-2x-2x^2+2x^3}$
$\{123, 132, 321\}$	0, for $n \geq 5$
$\{123, 213, 312\}$	$n+2$, for $n \geq 2$
$\{132, 213, 312\}$	
$\{132, 213, 321\}$	n , for $n \geq 3$
$\{132, 312, 321\}$	$4(n-1)$, for $n \geq 2$
$\{123, 132, 213, 321\}$	0, for $n \geq 5$
$\{123, 132, 231, 321\}$	
$\{123, 132, 312, 321\}$	
$\{123, 132, 213, 231\}$	4, for $n \geq 2$
$\{123, 132, 231, 312\}$	2, for $n \geq 3$
$\{132, 213, 231, 312\}$	
$\{123, 132, 213, 231, 312\}$	1, for $n \geq 3$
$\{123, 132, 213, 231, 321\}$	0, for $n \geq 5$

Table 2: A summary of the enumeration of nonnesting permutations avoiding subsets of $\mathcal{S}_3$ of size at least 2.

Λ	Formula for $s_n(\Lambda)$
$\{1123, 3211\}$	0, for $n \geq 5$
$\{1123, 3321\}$	
$\{1123, 1132\}$	2^{n-1}Cat_n
$\{2113, 3112\}$	
$\{1123, 1132, 2311\}$	$\frac{1}{6}(n-1)(n)(n+10)$, for $n \geq 2$
$\{1132, 2113, 2311\}$	$\frac{1}{6}(n^3 + 6n^2 - 7n + 6)$
$\{1132, 3112, 3121\}$	$5 \cdot 3^{n-2} - 1$, for $n \geq 2$
$\{1231, 1321, 2132, 2312, 3123, 3213\}$	$n!F_{n+1}$

Table 3: A summary of the enumeration of nonnesting permutations avoiding other patterns.

Λ	Conjecture for $s_n(\Lambda)$
$\{1231, 1321\}$	$\sum_{k=0}^{\infty} 2^{n+1} \frac{k^n}{3^{k+1}}$
$\{1231, 3213\}$	e.g.f.: $\frac{1}{\cos(\log(1-x)) + \sin(\log(1-x))}$
$\{1231, 1321, 2113\}$	$\sum_{k=0}^{n+1} 3^k \frac{1}{n} \binom{n}{k} \binom{n}{k+1}$
$\{1231, 3213, 3123\}$	e.g.f.: $\frac{1}{3+x-2e^x}$
$\{1231, 1312, 2231, 3221\}$	o.g.f.: $\frac{1-3x+2x^2}{(1-3x)(1-x-x^2)}$
$\{1231, 1213, 2132, 2231\}$	$\frac{11 \cdot 4^{n-2} + 1}{3}$
$\{1312, 2132, 2213, 3231\}$	o.g.f.: $\frac{1}{2x} + x - \frac{\sqrt{1-4x-4x^2+4x^4}}{2x}$
$\{1123, 2113, 2311\}$	$\lceil 3(n-1) \log(3n-3) \rceil$, for $n \geq 2$

Table 4: Conjectures of the enumeration of nonnesting permutations avoiding other patterns.

REFERENCES

[1] Kassie Archer, Adam Gregory, Bryan Pennington, and Stephanie Slayden. Pattern restricted quasi-Stirling permutations. *Australas. J. Combin.*, 74:389–407, 2019.

[2] Ira M. Gessel and Richard P. Stanley. Stirling polynomials. *J. Combinatorial Theory Ser. A*, 24(1):24–33, 1978.

[3] Sergi Elizalde. Descents on nonnesting multipermutations *European Journal of Combinatorics*, 103846, 2023.

The linear span of all words (that is, finite sequences of positive integers) has a bialgebra structure in which the product is a shifted shuffling operation and the coproduct corresponds to deconcatenation. Given a relation on words, one can consider the subspace of this bialgebra spanned by its equivalence classes. This subspace is not always a sub-bialgebra, but we can exactly characterize the word relations for which this is the case. Many familiar bialgebras and Hopf algebras arise from this construction. They are each equipped with a canonical morphism to quasisymmetric functions, and we can further characterize when the image of this morphism lies in the subalgebra of symmetric functions. A notable example comes from the word relation generated by the braid relations for an infinite symmetric group. From this case, one obtains interesting bialgebras on permutations and on 321-avoiding permutations, whose product has a nontrivial but explicit formula.

CYCLIC SIEVING AND HOMOMESY ON PERMUTATIONS: AN ANALYSIS OF MAPS AND STATISTICS IN THE FINDSTAT DATABASE

Erin McNicholas Willamette University *This talk is based on joint work with Ashleigh Adams, Jennifer Elder, Nadia Lafrenie re, Jessica Striker, and Amanda Welch*

In this abstract we systematically study the dynamical phenomena of cyclic sieving and homomesy on permutations, highlighting the results most relevant to permutation patterns. This work is based on [5, 1].

The cyclic sieving phenomenon (CSP), defined by Reiner, Stanton, and White in 2004 [8], occurs for a set when evaluating a polynomial at particular roots of unity counts elements fixed under corresponding powers of an action. It has been observed on various objects, including tableaux, words, set partitions, paths, permutations, and Catalan objects (see e.g. [10]).

Definition 1 ([8]). Given a set S , a polynomial $f(q)$, and a bijective action g of order n , the triple $(S, f(q), g)$ exhibits the **cyclic sieving phenomenon (CSP)** if, for all d , $f(\zeta^d)$ counts the elements of S fixed under g^d , where $\zeta = e^{2\pi i/n}$.

For each of our results, the set considered is the set of permutations of $[n]$ and the polynomial is the generating function of a permutation statistic. Thus, instead of referring to such a triple, we say a given permutation *statistic* exhibits the CSP with respect to a given *map*.

Defined in 2015 by Propp and Roby [7], homomesy relates the average of a given statistic over some set to the averages over orbits formed by a bijective map.

Definition 2 ([7]). Given a finite set S , a bijective map $\mathcal{X} : S \rightarrow S$, and a statistic $f : S \rightarrow \mathbb{Z}$, we say that $(S, \mathcal{X}, f)$ exhibits **homomesy** if there exists $c \in \mathbb{Q}$ such that for every orbit $\mathcal{O}$, $\frac{1}{|\mathcal{O}|} \sum_{x \in \mathcal{O}} f(x) = c$.

Our approach to finding instances of the CSP and homomesy on permutations is as follows: rather than test specific statistics that arise in other research, we use SageMath [11] to search the online combinatorial statistics database FindStat [9] for apparent instances of these phenomena. In [5], we identified and proved 122 instances of the homomesy phenomenon. In [1], we proved 34 instances of the CSP and conjectured 3 more. We refer to the maps and statistics by name and number from the [Findstat.org](https://findstat.org) database.

Cyclic sieving results [1]

Definition 3. A **Mahonian** statistic is a permutation statistic whose generating function is:

$$\sum_{\sigma \in S_n} q^{\text{Statistic}(\sigma)} = (1+q)(1+q+q^2) \dots (1+q+\dots+q^{n-1}) =: [n]_q!.$$

The inversion number and major index are Mahonian statistics. Several other statistics, stat , mak , mad , stat' , stat'' , makl , and hag , all appear in the work of Babson and Steingrímsson on the translation of Mahonian statistics into permutation patterns, where they are shown to be Mahonian [3].

The inversion number statistic under rotation is one of the original instances of the CSP proved in [8, Theorem 1.1]. We show several other maps exhibit the CSP with Mahonian statistics.

Theorem 4. *The Mahonian statistics exhibit the cyclic sieving phenomenon under any map whose orbits all have size d , where $1 \leq d \leq n$. Such maps include reverse, complement, toric promotion, and rotation.*

Reverse and complement (Maps 64 and 69)

Theorem 5. *The reverse and complement maps exhibit the CSP with 29 statistics, including:*

- *Statistic 7: The number of saliances (right-to-left maxima),*
- *Statistic 31: The number of cycles in the cycle decomposition,*
- *Statistic 538: The number of even inversions,*
- *Statistic 638: The number of up-down runs,*
- *Statistic 1579: The number of cyclically simple transpositions needed to sort a permutation,*
- *Statistic 1115: The number of even descents,*
- *Statistic 423: The number of occurrences of the pattern 123 or of the pattern 132,*
- *Statistic 494: The number of inversions of distance at most 3 when n is odd. (We leave as a conjecture that this also holds for n even.)*

The Corteel map (Map 239)

The Corteel map was constructed in [4] to interchange the number of crossings and the number of nestings of a permutation, giving a combinatorial proof of their equidistribution. We show this involution has 2^{n-1} fixed points, thus the CSP holds when the evaluation of the statistic generating function at -1 equals 2^{n-1} . We also show several equidistribution results; statistics are grouped by distribution in the theorem below.

Theorem 6. *The following statistics exhibit the CSP with respect to the Corteel map.*

- *Statistic 39: The number of crossings, Statistic 223: The number of nestings, and Statistic 356: The number of occurrences of the pattern 13 – 2.*

- *Statistic 317: The cycle descent number and Statistic 1744: The number of occurrences of the arrow pattern $\frac{12}{1 \rightarrow 2}$.*
- *Statistic 371: The number of midpoints of decreasing subsequences of length 3, Statistic 1683: The number of distinct positions of the pattern letter 3 in occurrences of 132, and Statistic 1004: The number of indices that are left-to-right maxima or right-to-left minima.*
- *Statistic 357: The number of occurrences of the pattern $12 - 3$.*

We also conjecture that Statistic 373 (the number of weak exceedances that are also mid-points of a decreasing subsequence of length 3) exhibits the CSP with respect to the Corteel map.

Alexandersson-Kebede map (Map 257)

The Alexandersson-Kebede map κ is an involution on permutations that preserves right-to-left minima. It was introduced in [2] to give a bijective proof of an identity regarding derangements and to refine it with respect to right-to-left minima.

Definition 7. Let $\sigma \in S_n$. The **Alexandersson-Kebede map** κ is defined as follows. Let i be the smallest odd integer such that $\sigma \cdot (i, i + 1)$ and σ have the same set of right-to-left minima, if such an i exists. If it does not, then $\kappa(\sigma) = \sigma$. Otherwise, $\kappa(\sigma) = \sigma \cdot (i, i + 1)$.

Example 8. Consider the permutation $\sigma = 2134756$. Its right-to-left minima are $\{1, 3, 4, 5, 6\}$. To find $\kappa(\sigma)$, we need to find the smallest odd i for which $\sigma \cdot (i, i + 1)$ and σ have the same right-to-left minima. We cannot choose $i = 1$ since $\sigma \cdot (1, 2) = 1234756$ makes 2 a right-to-left minimum. We cannot choose $i = 3$ since $\sigma \cdot (3, 4) = 2143756$ makes 4 not a right-to-left minimum anymore. Then $i = 5$ since 2134756 and 2134576 have the same set of right-to-left minima. Here, $\kappa(\sigma) = 2134576$.

Theorem 9. *Statistic 1005: The number of indices that are l-to-r maxima or r-to-l minima but not both, and Statistic 1727: The number of invisible inversions, both exhibit the CSP with respect to the Alexandersson-Kebede map.*

Homomesy results [5]

Lehmer code rotation (Map 149)

Definition 10. The **Lehmer code** of a permutation $\sigma \in S_n$ is: $L(\sigma) = (L(\sigma)_1, \dots, L(\sigma)_n)$ where $L(\sigma)_i = \#\{j > i \mid \sigma_j < \sigma_i\}$. The **Lehmer code rotation** is a map $\mathbb{L}$ that sends a permutation σ to the unique permutation τ (of the same set) such that every entry in the Lehmer code of τ is cyclically (modulo $n + 1 - i$) one larger than the Lehmer code of σ .

Example 11. The permutation $\sigma = 31452$ has Lehmer code $L(\sigma) = (2, 0, 1, 1, 0)$. Hence,

$$\begin{aligned} L(\mathbb{L}(\sigma)) &= (2+1 \mod 5, 0+1 \mod 4, 1+1 \mod 3, 1+1 \mod 2, 0+1 \mod 1) \\ &= (3, 1, 2, 0, 0). \end{aligned}$$

Because $(3, 1, 2, 0, 0)$ is the Lehmer code of the permutation 42513, $\mathbb{L}(31452) = 42513$.

Theorem 12. *The Lehmer code rotation map exhibits homomesy for 45 FindStat statistics, including:*

- Statistic 18: *The number of inversions (average: $\frac{n(n-1)}{4}$)*
- Statistic 495: *The number of inversions of distance at most 2 (average: $\frac{2n-3}{2}$)*
- Statistic 21: *The number of descents (average: $\frac{n-1}{2}$)*
- Statistic 470: *The number of runs (average: $\frac{n+1}{2}$)*
- Statistic 356: *The number of occurrences of the pattern $13-2$ (average: $\frac{(n-1)(n-2)}{12}$)*
- Statistic 423: *The number of occurrences of the pattern 123 or of the pattern 132 (average: $\frac{1}{3}\binom{n}{3}$)*
- Statistic 709: *The number of occurrences of $14-2-3$ or $14-3-2$ (average: $\frac{1}{12}\binom{n-1}{3}$)*
- Statistic 1084: *The number of occurrences of the vincular pattern $|1-23$ (average: $\frac{n-2}{6}$)*
- Statistic 7: *The number of saliances (right-to-left maxima) (average: $H_n = \sum_{i=1}^n \frac{1}{i}$)*
- Statistic 692: *Babson and Steingrímsson's statistic stat (average: $\frac{n(n-1)}{4}$)*

Reverse and Complement (Maps 64 and 69)

Theorem 13. *The reverse map and the complement map are both homomesic under 22 statistics, including:*

- Statistic 18: *The number of inversions (average: $\frac{n(n-1)}{4}$)*
- Statistic 21: *The number of descents (average: $\frac{n-1}{2}$)*
- Statistic 457: *The number of occurrences of one of the patterns 132, 213, or 321 (average: $\frac{\binom{n}{3}}{2}$)*

The complement map, but not the reverse, is homomesic under 13 statistics, including:

- Statistic 4: *The major index (average: $\frac{n(n-1)}{4}$)*
- Statistic 692: *Babson and Steingrímsson's statistic stat (average: $\frac{n(n-1)}{4}$)*

The reverse map, but not the complement, is homomesic under 5 statistics, including:

- Statistic 305: The inverse major index (average: $\frac{n(n-1)}{4}$)
- Statistic 798: The *makl* (average: $\frac{n(n-1)}{4}$)

Kreweras complement (Map 88)

The Kreweras complement was introduced in 1972 as a bijection on noncrossing partitions [6] and may be extended to all permutations as follows.

Definition 14. Let σ be a permutation of n elements. The **Kreweras complement** of σ is defined as $\mathcal{K}(\sigma) = c \circ \sigma^{-1}$ where c is the long cycle $234 \dots 1$.

Theorem 15. The Kreweras complement exhibits homomesy for 4 statistics, including Statistic 155: The number of exceedances of a permutation (average: $\frac{n-1}{2}$), and Statistic 702: The number of weak deficiencies of a permutation (average: $\frac{n+1}{2}$).

REFERENCES

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- [1] Ashleigh Adams, Jennifer Elder, Nadia Lafrenière, Erin McNicholas, Jessica Striker, and Amanda Welch. Cyclic sieving on permutations – an analysis of maps and statistics in the findstat database. arXiv:2402.16251.
 - [2] Per Alexandersson and Frether Getachew. An involution on derangements preserving excedances and right-to-left minima. *Australasian Journal of Combinatorics*, 86(3):387–413, 2023.
 - [3] Eric Babson and Einar Steingrímsson. Generalized permutation patterns and a classification of the Mahonian statistics. *Sém. Lothar. Combin.*, 44:Art. B44b, 18 pages, 2000.
 - [4] Sylvie Corteel. Crossings and alignments of permutations. *Advances in Applied Mathematics*, 38(2):149–163, 2007.
 - [5] Jennifer Elder, Nadia Lafrenière, Erin McNicholas, Jessica Striker, and Amanda Welch. Homomesies on permutations: An analysis of maps and statistics in the findstat database. *Mathematics of Computation*, 2023.
 - [6] Germain Kreweras. Sur les partitions non croisées d’un cycle. *Discrete mathematics*, 1(4):333–350, 1972.
 - [7] James Propp and Tom Roby. Homomesy in products of two chains. *Electron. J. Combin.*, 22(3):Paper 3.4, 29 pages, 2015.
 - [8] V. Reiner, D. Stanton, and D. White. The cyclic sieving phenomenon. *Journal of Combinatorial Theory, Series A*, 108(1):17-50, 2004.

- [9] Martin Rubey, Christian Stump, et al. FindStat - The combinatorial statistics database. <http://www.FindStat.org>. Accessed: April 27, 2022.
- [10] Bruce E. Sagan. The cyclic sieving phenomenon: a survey. In *Surveys in combinatorics 2011*, volume 392 of *London Math. Soc. Lecture Note Ser.*, pages 183–233. Cambridge Univ. Press, Cambridge, 2011.
- [11] William A. Stein et al. Sage Mathematics Software (Version 9.4). The Sage Development Team, 2022. <http://www.sagemath.org>.

THE HIERARCHY OF HEREDITARY SORTING OPERATORS

Michal Opler Faculty of Information Technology, Czech Technical University in Prague

This talk is based on joint work with Vít Jelínek and Jakub Pekárek

We consider the following general model of a sorting procedure: we fix a permutation class $\mathcal{C}$, which corresponds to the operations that the procedure is allowed to perform in a single step. The input of sorting is a permutation π of the set $[n] = \{1, 2, \dots, n\}$, i.e., a sequence where each element of $[n]$ appears once. In every step, the sorting procedure picks a permutation σ of length n from $\mathcal{C}$, and rearranges the current permutation of numbers by composing it with σ . The goal is to transform the input π into the sorted sequence $1, 2, \dots, n$ in as few steps as possible.

Formally, for a permutation class $\mathcal{C}$ and a permutation π of $[n]$, we say that $\mathcal{C}$ can sort π in k steps, if the inverse of π can be obtained by composing k (not necessarily distinct) permutations from $\mathcal{C}$. The $\mathcal{C}$ -sorting time of π , denoted $\text{st}(\mathcal{C}; \pi)$, is the smallest k such that $\mathcal{C}$ can sort π in k steps; if no such k exists, we put $\text{st}(\mathcal{C}; \pi) = +\infty$. For an integer n , the worst-case $\mathcal{C}$ -sorting time, denoted $\text{wst}(\mathcal{C}; n)$, is the maximum of $\text{st}(\mathcal{C}; \pi)$ over all permutations π of $[n]$.

This model of sorting captures not only classical sorting algorithms, like insertion sort or bubble sort, but also sorting by series of devices, like stacks or parallel queues, as well as sorting by block operations commonly considered, e.g., in the context of genome rearrangement.

Our goal is to describe the possible asymptotic behavior of the function $\text{wst}(\mathcal{C}; n)$, and relate it to structural properties of $\mathcal{C}$. As the main result, we show that any hereditary permutation class $\mathcal{C}$ falls into one of the following five categories:

- $\text{wst}(\mathcal{C}; n) = +\infty$ for n large enough,
- $\text{wst}(\mathcal{C}; n) = \Theta(n^2)$,
- $\Omega(\sqrt{n}) \leq \text{wst}(\mathcal{C}; n) \leq O(n)$,
- $\Omega(\log n) \leq \text{wst}(\mathcal{C}; n) \leq O(\log^2 n)$, or
- $\text{wst}(\mathcal{C}; n) = 1$ for all $n \geq 2$.

In addition, we characterize the permutation classes in each of the five categories.

This talk is based on joint work with Andrei Asinowski²

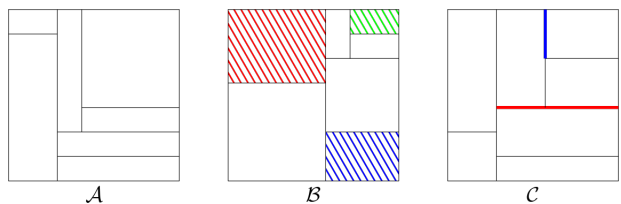
Abstract. We consider rectangulations that avoid patterns from the set $\{\top, \perp, \vdash, \dashv\}$. We present enumerative results, as well as bijections to pattern-avoiding inversion sequences and families of Dyck paths.

Introduction. A *rectangulation* $\mathcal{R}$ is a partition of a rectangle R into a finite number of rectangles such that there are no $+$ joints. The *size* of $\mathcal{R}$ is the number of interior rectangles, and the corners of $\mathcal{R}$ are labeled by compass directions. A *segment* in $\mathcal{R}$ is a maximal union of edges of rectangles which form a straight line (but not an edge of R). The *neighbors* of a segment c are the perpendicular segments which have an endpoint on c . In the figure below, the red line is a segment, while the blue line is not. See [2, 5] for links between rectangulations and other combinatorial objects such as lattice paths, binary trees, and permutations.

Within a rectangulation $\mathcal{R}$, rectangle r is a *right neighbor* of rectangle s if there exists a segment containing the right edge of s and the left edge of r . Rectangle r is *right of* rectangle t if there exist a sequence of rectangles beginning with t and ending with r such that each rectangle is a right neighbor of the previous one. Above and below are defined analogously. In rectangulation $\mathcal{B}$, the blue rectangle is a right neighbor of the red one, both the green and blue rectangles are right of the red one, and the blue rectangle is below the green one.

There are two natural ways to define combinatorial equivalence for rectangulations: *weak equivalence* preserves left-right and up-down relations between rectangles, while *strong equivalence* also preserves contact between rectangles. Equivalence classes are called *weak rectangulations* and *strong rectangulations* respectively. For example, rectangulations $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are all weakly equivalent, but only $\mathcal{B}$ and $\mathcal{C}$ are strongly equivalent.

A rectangulation *avoids* $\top$ if there is no $\top$ joint within it (defined analogously for $\perp$, $\vdash$, and $\dashv$). The property of avoiding these patterns is invariant to both kinds of equivalence. For example, all the rectangulations in the drawing avoid $\top$.



An integer sequence $s = (s_1, s_2, \dots, s_n)$ is an *inversion sequence* if for each $1 \leq i \leq n$, we have $0 \leq s_i \leq i - 1$. We can naturally plot s on an $n \times n$ grid so that all points lie on

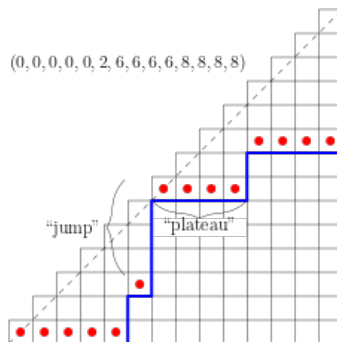
¹Michaela Polley is supported by a Fulbright-Austrian Marshall Plan Foundation Award For Research In Science And Technology.

²Andrei Asinowski is supported by FWF — Austrian Science Fund, grant P32731.

Our results. We study pattern avoidance in rectangulations, mainly enumeration of avoidance classes and links to pattern avoidance in other combinatorial structures. The systematic study of pattern avoidance in rectangulations was started by Merino and Mütze in [7], where eight patterns were considered (see also [1] where several models involving these patterns were solved). In the current work, we consider the pattern $\top$ and its rotations, taken in all possible combinations. Note that these are the most simple patterns, and every other pattern can be built from them. Hence, results concerning avoidance of $\top$ can be foundational for general results on pattern avoidance in rectangulations.

Theorem 1 (Enumerating $R_n^w(\top)$). *We have $|R_n^w(\top)| = C_n$, the n^{th} Catalan number.*

8	6	2	0
8			0
8	6	0	
8			
6			0
6			0

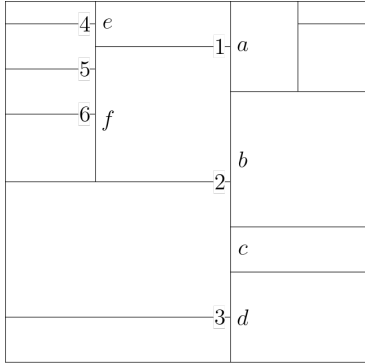
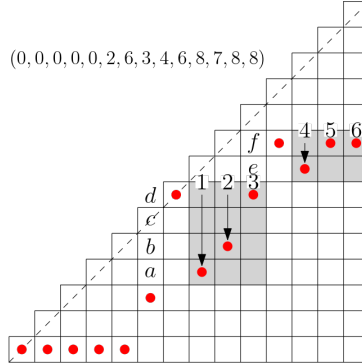


³A different proof of Theorem 1 was given by Aaron Williams (personal communication).

Starting with a non-decreasing inversion sequence, the weak rectangulation is constructed by considering each “plateau” (a subsequence whose elements all have the same value) and “jump” (the vertical distance between plateaus), where each plateau corresponds to a vertical segment in or the right edge of the rectangulation. Non-decreasing inversion sequences are in bijection with Dyck paths, as shown in blue in the figure. $\square$

Theorem 2 (Enumerating $R_n^s(\top)$). *We have $|R_n^s(\top)| = |I_n(110, 210, 010, 120)|$.*

Proof. Since we are treating $\mathcal{R}$ as a strong rectangulation, we must account for the contact between rectangles, or equivalently, the order of neighbors along each vertical segment. Begin by constructing a non-decreasing inversion sequence as in Theorem 1. Each vertical segment corresponds to a jump followed by a plateau, where the size of the jump is one more than the number of neighbors to its right (the number of $\vdash$ joints), and the length of the plateau is one more than the number of neighbors to its left (the number of $\dashv$ joints).



For each vertical segment, label the left neighbors from top to bottom with numerals and the regions above, below, and between the right neighbors with letters. Similarly, label the corresponding plateau starting with the second element and moving to the right with numerals and the jump with letters starting at the bottom.

Then, each left neighbor corresponds to an element of the plateau and it is shifted down as necessary to match which region it is in.

The resulting inversion sequence belongs to $I_n(110, 210, 010, 120)$, and this mapping is bijective. This class of inversion sequences was considered by Martinez and Savage [6], its enumerative sequence is $\square$.

Proposition 3 (Enumerating $R_n(\vdash, \dashv)$). (a) *We have $|R_n^w(\vdash, \dashv)| = 2^{n-1}$.* (b) *There is a bijection between $R_n^s(\vdash, \dashv)$ and rushed paths — a family of Dyck paths defined below — of length $2(n+1)$.*

Proof. A $(\vdash, \dashv)$ -avoiding rectangulation can be obtained by drawing $k \geq 0$ horizontal

segments, and then adding ≥ 0 vertical segments in thus obtained $k + 1$ rectangles. (a) As weak rectangulations, this structure is equivalent to compositions of n . Accordingly, the generating function is readily seen to be $\frac{1}{1-\frac{1}{1-x}}$. (b) A *rushed path* is a Dyck path that starts with $h \geq 1$ up-steps, and then never reaches the height h again. Given

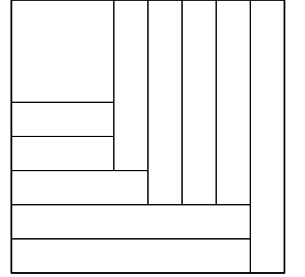
a rushed path P of length $2(n + 1)$, we consider the grid rectangle $[h + 1, 2(n + 1)] \times [0, h - 1]$, draw horizontal grid lines to form $h - 1$ rectangles, and replace every up-step of P (within the rectangle) by a vertical segment of height 1. In



this way we obtain a strong $(\vdash, \dashv)$ -avoiding rectangulation of size n , and this mapping is a bijection. In the drawing, a rushed path is black, and the corresponding rectangulation is red. The enumerative sequence is , where a formula, as well as asymptotics recently obtained by Bacher [3], is given. $\square$

Proposition 4 (Enumerating $R_n(\top, \vdash)$). We have $|R_n(\top, \vdash)| = 2^{n-1}$.

Proof. Using the same bijection to non-decreasing inversion sequences as in the proof of Theorem 1, a $(\top, \vdash)$ -avoiding rectangulation will map to an inversion sequence in which each jump has size exactly one. Such an inversion sequence has $n - 1$ pairs of adjacent elements, each of which either has both elements with the same value, or the second element will be one greater than the first element. Hence, there are 2^{n-1} such sequences. Alternatively, we can build such a rectangulation by starting with a single rectangle and adding $n - 1$ rectangles sequentially, each time either to the right of or below all existing rectangles. $\square$



Observation 5 (Enumerating $R_n(\top, \perp, \vdash)$, $R_n(\top, \perp, \vdash, \dashv)$). (a) $|R_n(\top, \perp, \vdash)| = n$. (b) $|R_n(\top, \perp, \vdash, \dashv)| = 2$.

Proof. (a) A rectangulation in $R_n(\top, \perp, \vdash)$ can be obtained by drawing k vertical segments ($0 \leq k \leq n - 1$), and then adding $n - k - 1$ horizontal segments in the leftmost of thus obtained $k + 1$ rectangles. It is completely determined by the choice of k , hence, there are n such rectangulations. (b) A rectangulation in $R_n(\top, \perp, \vdash, \dashv)$ has either all vertical or all horizontal segments. There are only two such rectangulations. $\square$

REFERENCES

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- [1] Andrei Asinowski and Cyril Banderier. [arXiv:2401.05558](#) (2024).
 - [2] Andrei Asinowski, Jean Cardinal, Stefan Felsner, and Éric Fusy. [arXiv:2402.01483](#) (2024).
 - [3] Axel Bacher. [arXiv:2403.08120](#) (2024).

- [4] Sylvie Corteel, Megan A. Martinez, Carla D. Savage, and Michael Weselcouch. [Discrete Mathematics and Theoretical Computer Science](#), 18 (2016).
- [5] Stefan Felsner, Éric Fusy, Marc Noy, and David Orden. [Journal of Combinatorial Theory, Series A](#), 118:3 (2011), 993–1020.
- [6] Megan A. Martinez and Carla D. Savage. [Journal of Integer Sequences](#), 21 (2018), Article 18.2.2.
- [7] Arturo Merino and Torsten Mütze. [Discrete and Computational Geometry](#), 70 (2023), 51–122.

When researchers communicate with researchers, we identify “the next natural questions” to pursue. Through this process, we’ve seen multiple threads of permutation pattern research arise over the two decades of this conference, for example, progressing from questions of exact enumeration to probabilistic results.

But what happens when researchers communicate with other audiences who are eager to engage with the content, but are not themselves researchers? This talk will explore results that have arisen from conveying permutation patterns in non-research contexts, including public lectures and math camps. In particular we’ll explore questions — some solved, some open — that have arisen by presenting pattern-avoidance definitions and theorems in the form of games.

This talk is based on joint work with Valentin Féray

We will present a general approach for constructing a permuton-valued process with a given stationary distribution. This target stationary distribution must be a weak limit of probability distributions on permutations of increasing sizes $M_1, M_2, M_3, \dots$ that satisfy a certain symmetry condition and the following (i) *recursive* and (ii) *consistency* properties:

- (i) for each n , there is a transition matrix U_n satisfying $M_n U_n = M_{n+1}$, and
- (ii) if σ_{n+1} is a random permutation with distribution M_{n+1} , then the permutation obtained by removing a uniformly random point from σ_{n+1} has distribution M_n .

These two properties allow us to first address the question at the discrete level – that is, to construct Markov chains for which the $\{M_n\}$ are stationary. Indeed, if we let D_{n+1} be the transition matrix that removes a uniformly random point from a permutation of $[n+1]$, then the transition matrices $T_n = U_n D_{n+1}$ will give rise to such chains.

We then move from these permutation chains to the permuton process by taking a suitable limit. Here, we rely heavily on our symmetry condition, a commutation relation satisfied by the $\{U_n\}$ and $\{D_n\}$. This condition will ensure the convergence of the chains and allow us to perform explicit computations throughout. Most notably, it will enable the convergence argument to be tackled by essentially algebraic methods.

Through this limit, we will obtain an ergodic Feller process (in particular, a strong Markov process). It will have the desired distribution (the limit of the $\{M_n\}$) as a unique stationary distribution and will exhibit a number of additional properties. For example, its generator will admit a simple description in terms of pattern density functions and an explicit diagonal description.

To illustrate our framework, we will discuss an example involving separable permutations. Here, the growth dynamics (i.e. the $\{U_n\}$) enlarge permutations by creating new adjacencies. The $\{M_n\}$ are the distributions of the *recursive separable permutations* that we introduced in [3]. In that paper, we showed that these permutations converge to a random permuton that we called the *recursive separable permuton*. An application of our theory thus provides a process that realizes this permuton in stationarity.

If time permits, we will discuss some broader interpretations of our work that extend beyond the setting of permutations and permutons. This includes connections to the work of Fulman [4], Borodin and Olshanski [1], and Petrov [5] on *up-down chains* and the work of Borodin and Olshanski [2] on the *method of intertwiners*.

REFERENCES

- [1] Alexei Borodin and Grigori Olshanski. Infinite-dimensional diffusions as limits of random walks on partitions. *Probability Theory and Related Fields*, 144(1):281–318, 2009.
- [2] Alexei Borodin and Grigori Olshanski. Markov processes on the path space of the Gelfand–Tsetlin graph and on its boundary. *Journal of Functional Analysis*, 263:248–303, 2012.
- [3] Valentin Féray and Kelvin Rivera-Lopez. The permuton limit of random recursive separable permutations. 2023.
- [4] Jason Fulman. Commutation relations and Markov chains. *Probability Theory and Related Fields*, 144(1):99–136, 2009.
- [5] L. A. Petrov. Two parameter family of infinite-dimensional diffusions on the Kingman simplex. *Functional Analysis and its Applications*, 43(4):279–296, 2009.

This talk is based on joint work with Mark Dukes

For n a nonnegative integer we let $[n] = \{1, 2, \dots, n\}$. Consider a sequence $\alpha = a_1 a_2 \dots a_n$ of nonnegative integers where we will sometimes insert commas between the elements for readability. Note that we will use Greek letters for sequences and the corresponding roman letters for their elements. For $k \in [n]$ we will use

$$\alpha_k = a_1 a_2 \dots a_k$$

for the prefix of α with k elements. The *ascent set* and *ascent number* of α are

$$\text{Asc } \alpha = \{k \in [n-1] \mid a_{k+1} > a_k\}$$

and

$$\text{asc } \alpha = \# \text{Asc } \alpha,$$

respectively, where we will use a hash symbol to denote cardinality. For example, if

$$\alpha = 0, 1, 1, 0, 2, 1, 2, 4 \tag{5}$$

then $\text{Asc } \alpha = \{1, 4, 6, 7\}$ and $\text{asc } \alpha = 4$. Call α an *ascent sequence* if

(a1) $a_1 = 0$, and

(a2) $a_{k+1} \leq 1 + \text{asc } \alpha_k$ for $k \in [n-1]$.

It is easy to check that α in the previous example is an ascent sequence while

$$\alpha' = 0, 1, 1, 0, 3, 1, 2, 4 \tag{6}$$

is not because $a'_5 = 3$ while $1 + \text{asc}(0, 1, 1, 0) = 2$ which is smaller. For $n \geq 0$ we will use the notation

$$A_n = \{\alpha = a_1 a_2 \dots a_n \mid \alpha \text{ is an ascent sequence}\}.$$

For example

$$A_3 = \{000, 001, 010, 011, 012\}.$$

Bousquet-Mélou, Claesson, Dukes and Kitaev [BMCDK10] were the first to define and study ascent sequences and they have since been considered by numerous authors. Ascent sequences are known to be in bijection with unlabeled $(2+2)$ -free posets, permutations avoiding a bivincular pattern of length 3, certain upper-triangular nonnegative integer matrices, and matchings with no neighbor nestings.

Very recently, Bényi, Claesson, and Dukes [BCD23] introduced weak ascent sequences. The set of *weak ascents* of α is

$$\text{wAsc } \alpha = \{k \in [n-1] \mid a_{k+1} \geq a_k\}$$

with corresponding *weak ascent number*

$$\text{wasc } \alpha = \# \text{wAsc } \alpha.$$

Note that $\text{wAsc } \alpha \supseteq \text{Asc } \alpha$. Taking α' as in (6) we see that $\text{wAsc } \alpha' = \{1, 2, 4, 6, 7\}$. The sequence α is a *weak ascent sequence* if

(w1) $a_1 = 0$, and

(w2) $a_{k+1} \leq 1 + \text{wasc } \alpha_k$ for $k \in [n - 1]$.

Even though α' as above was not an ascent sequence, it is a weak ascent sequence. In fact, if we let

$$\text{wA}_n = \{\alpha = a_1 a_2 \dots a_n \mid \alpha \text{ is a weak ascent sequence}\}.$$

then clearly $\text{wA}_n \supseteq \text{A}_n$ for all $n \geq 0$. As an example,

$$\text{wA}_3 = \{000, 001, 002, 010, 011, 012\}.$$

In [BCD23] the authors showed that weak ascent sequences are equinumerous with factorial posets avoiding a specially labeled $3 + 1$ poset, permutations avoiding a bivincular pattern of length 4, upper-triangular 0-1 matrices satisfying a column restriction, and matchings with no left-neighbor nestings.

After Dukes presented his work at Permutation Patterns 2023, Sagan proposed the following generalization of both ascent sequences and weak ascent sequences. For $d \geq 0$ the *difference d ascent set* or simply *d -ascent set* of α is

$$\text{dAsc } \alpha = \{k \in [n - 1] \mid a_{k+1} > a_k - d\}.$$

with corresponding *d -ascent number*

$$\text{dasc } \alpha = \# \text{dAsc } \alpha.$$

Note that when $d = 0$ and $d = 1$ we have $\text{dAsc } \alpha = \text{Asc } \alpha$ and $\text{dAsc } \alpha = \text{wAsc } \alpha$, respectively. For example, when $d = 2$ we have

$$\text{dAsc}(0, 1, 0, 2, 3, 1, 4, 2, 6) = \{1, 2, 3, 4, 6, 8\}.$$

Unsurprisingly, the definition of a *d -ascent sequence* α is one where

(d1) $a_1 = 0$, and

(d2) $a_{k+1} \leq 1 + \text{dasc } \alpha_k$ for $k \in [n - 1]$.

The example sequence just given is a 2-ascent sequence. Let

$$\text{dA}_n = \{\alpha = a_1 a_2 \dots a_n \mid \alpha \text{ is a } d\text{-ascent sequence}\}.$$

In this talk we will show how the bijections in the ascent and weak ascent cases can be generalized to any d . In some cases, the maps become injective rather than bijective which raises the open problem of how to restrict the range of the function in a natural way so as to obtain a bijection. We will concentrate on the relationship between d -ascent sequences and permutations avoiding a particular bivincular pattern of length $d + 3$. More details can be found in the preprint [DS23].

REFERENCES

- [BCD23] Beáta Bényi, Anders Claesson, and Mark Dukes. Weak ascent sequences and related combinatorial structures. *European J. Combin.*, 108:Paper No. 103633, 19, 2023.
- [BMCDK10] Mireille Bousquet-Mélou, Anders Claesson, Mark Dukes, and Sergey Kitaev. $(2 + 2)$ -free posets, ascent sequences and pattern avoiding permutations. *J. Combin. Theory Ser. A*, 117(7):884–909, 2010.
- [DS23] Mark Dukes, and Bruce Sagan. Difference ascent sequences. arXiv:2311.15370, 2023.

ASYMPTOTIC DICHOTOMY OF ALTERNATING SIGN MATRICES AVOIDING A PERMUTATION

Justin M. Troyka

California State University, Los Angeles

This talk is based on joint work with Mathilde Bouvel, Rebecca Smith, and Jessica Striker

Recall that an *alternating sign matrix* (ASM) is a matrix with entries from $\{0, 1, -1\}$ such that (1) every row and column begins and ends with 1 and (2) the non-zero entries in each row or column alternate between 1 and -1 . These are a generalization of permutations, since every permutation matrix is an ASM.

Let π be a size- n permutation, and let B be an alternating sign matrix (ASM). We say π is *contained in* B if rows and columns can be deleted from B to obtain an $n \times n$ matrix with 1's in at least the positions where the permutation matrix of π has 1's. As always, we say B *avoids* π if π is not contained in B . This matches the definition in Johansson and Linusson [2], who initiated the enumeration of ASMs avoiding a given permutation or set of permutations. Their main result is that the ASMs avoiding 132 are counted by the Schröder numbers. Other notions of pattern avoidance in ASMs have been studied, e.g. by Aval [1], who studies the number of ASMs B such that $\text{key}(B)$ avoids π , where key is a certain map from ASMs to permutations.

We prove general results on the asymptotic number of alternating sign matrices (ASMs) avoiding a given permutation pattern. Our main result is a dichotomy between classes that are at most exponential and classes that are at least $\lfloor n/3 \rfloor!$, as follows.

Theorem 1. *Let σ be a permutation. If σ contains 123 or 321, i.e. if*

$$\sigma \notin \{\varepsilon, 1, 12, 21, 132, 213, 231, 312, 2143, 2413, 3142, 3412\},$$

then the number of $n \times n$ ASMs avoiding σ is at least $\lfloor n/3 \rfloor!$ for all n .

That is, every class of ASMs avoiding a given permutation has super-exponential growth, the only possible exceptions being the twelve listed permutations which avoid both 123 and 321.

The idea of the proof is that, if π is an arbitrary permutation and A is its matrix, then every ASM of the form

$$\left[\begin{array}{c|cc} & 1 & \ddots & \\ \hline & & & 1 \\ \hline 1 & & & \\ & \ddots & & \\ & & -A & \\ & & & 1 \\ \hline & 1 & \ddots & \\ & & & 1 \end{array} \right]$$

avoids σ if σ contains 321. The number of $n \times n$ ASMs of this form is $\lfloor n/3 \rfloor!$, since π can be any permutation of size $\lfloor n/3 \rfloor!$.

As for the enumeration of the twelve possible exceptions, the ones of size ≤ 2 have trivial enumeration, and the ones of size 3 are symmetries of each other and were shown by [2] to be counted by the Schröder numbers, which confirms that they are indeed exponential (with growth rate $3 + 2\sqrt{2}$). We prove an exponential upper bound for each of 2143 and 3412 (equivalent to each other by symmetry): the number of $n \times n$ ASMs avoiding 3412 is at most 62208^n (the base 62208 is probably not the best possible). But the asymptotic enumeration of each of 2413 and 3142 (also equivalent to each other by symmetry) remains open — we do not even know whether it is exponential or super-exponential.

A natural open question: what more can we say about the asymptotic enumeration of σ -avoiding ASMs when σ is not one of our twelve possible exceptions? Is there a nice upper bound?

REFERENCES

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- [1] J.-C. Aval. Keys and alternating sign matrices. *Sém. Lothar. Combin.*, vol. 59 (2008), Article B59f.
 - [2] R. Johansson and S. Linusson. Pattern avoidance in alternating sign matrices. *Ann. Comb.*, vol. 11 (2007), p. 471–480.

Introduction

Permutations are typically defined as bijections on a set. In an n -dimensional vector space, there are $n!$ structurally independent permutations, forming an n -element symmetric group. The study of permutation patterns focuses on the pattern-avoidance problems, particularly emphasizing the avoidance of specific monotonicity, and exploring the relationships among these patterns. The objects and results typically are subsets of symmetric group, while most of permutations are excluded, just like the number of permutations without increasing is an Catalan number, much smaller than $n!$. Similarly, this paper investigates certain elements in the permutation group, but the focus is not on avoiding specific pattern but on linear independence (LI), LI permutations are called as isomers, with others being linear expression of isomers and thus discardable. This long abstract is divided into two parts: the first investigates the independence of permutations as mappings of vector fields, while the second explores the independence of permutations as transpositions of higher-order tensors. Each part briefly demonstrates the general formula for the amount of isomers and provides methods for constructing maximal isomer sets.

LI of Permutations as Mappings of Vector Fields

For vectors in n -dimensional space, the action of permutations takes the form of permutation matrices. It is evident that an n -dimension square matrix contains n^2 elements. the rank of the vector set formed by reshaping permutation matrices is no greater than n^2 . However, there $n!$ structurally distinct elements in an n -element permutation, each uniquely corresponding to a matrix, drawing the conclusion some permutations must be linearly dependent. For example, in a 3-element permutation, it is evident that for any 3D vector $\mathbf{X}$:

$$(\alpha_{123} + \alpha_{231} + \alpha_{312} - \alpha_{132} - \alpha_{213} - \alpha_{321} = 0)\mathbf{X} = \mathbf{0}$$

Indeed, the difference between all even permutations and odd permutations constitutes a zero mapping.

Through enumeration, we observed that all elements of unary and binary permutations are LI. In ternary permutations, any selection of 5 elements is LI, meaning that once 5 elements are chosen, the remaining elements can be uniquely linearly expressed. The number of isomers under matrix addition is denoted as $a_m(n)$, then $a_m(1) = 1$, $a_m(2) = 2$, and $a_m(3) = 5$. Next, we will employ a recursive approach to prove $a_m(n) = (n - 1)^2 + 1$, and develop a method for constructing independent set.

Suppose $a_m(n) = N$ and a set of n -element isomers is known as $\{\phi_1, \dots, \phi_N\}$. Then for any $(n+1)$ -dimensional vector $X_{(n+1)} = [x_1, x_2, \dots, x_{n+1}]^T$, the linear combination of the permutation set $\Phi_{n+1} := \{(\phi_i, n+1), i = 1, 2, \dots, N\}$ can map X_{n+1} to $[\sigma(X_n)^T, x_{n+1}]^T$ where σ represents any n -element permutation. These N permutations can linearly express all permutations with the last element fixed as x_{n+1} .

Φ_{n+1} cannot operate on x_{n+1} , so introduce a set of n perms $\Psi_{n+1} = \{\psi_i = (i, n+1), i = 1, 2, \dots, n\}$. It is obvious that Ψ_{n+1} is LI from Φ_{n+1} and each is linearly independent. The difference between the identity mapping (also a linear combination of elements in Φ_{n+1}) and ψ_i can exchange the positions of X_i and X_{n+1} . Thus, the extended set can permute X_{n+1} to more orders.

However, Φ_{n+1} and Ψ_{n+1} still cannot represent all permutations. Because in these sets, X_{n+1} is still constrained either to be at the end or to be transposed after being bound to ψ_i and X_i . It is also necessary to introduce an n order permutation set $\Omega_{n+1} = \{(2, 3, \dots, n+1, 1), (3, 4, \dots, n+1, 1, 2), \dots, (n+1, 1, 2, \dots, n)\}$, where one can be expressed by Φ_{n+1} and Ψ_{n+1} , the others are LI. Thus, the number of isomers is $n-1$, thus all permutations can be obtained through addition. Therefore, given the known values for $n = 1, 2, 3$ we have the formula:

$$a_m(n+1) - a_m(n) = 2n - 1$$

$$a_m(n) = (n-1)^2 + 1$$

A set of isomers can be obtained by selecting any 5 from $n = 3$ (the identity mapping is suggested) using the above method.

Independence of Permutations as Transposes of Tensors

Tensors, or multilinear mappings, are structures with multiple index notations. A vector X_i contains one index, a matrix Q_{ij} contains two indices, and a tensor $T_{i_1 i_2 \dots i_p}$ contains more than two indices. This paper assumes that each index represents spaces with the same dimension, denoted as n (where $i = 1, 2, \dots, n$), meaning n is the dimension of the base space and p is the order of the tensor. Tensors are an extension of square matrices to higher orders, and tensor transpose are defined as the permutation of the component indices. This section will investigate the linear independence of tensor transpositions and provide the number and methods of independent transpositions.

A p -order tensor in an n -dimensional space has n^p elements. And there are $p!$ permutations for p -element permutations, which is unconcerned with n . However, Smith[1] found that the number of $2p$ -order isotropic tensor isomers affected by n . Permutation tensor is a subset of $2p$ -isotropic tensor, who's dependency is also constrained by both p and n .

Developed by Robinson, Schensted and Knuth, the RSK algorithm [2, 3] demonstrates the equivalence between permutations and standard Young tableaux pairs. We have

found the shape of Young tableaux pairs corresponding to the transpose of p -order tensors, explaining the restriction of n , the dimension of the base space, on Young tableaux pairs: the number of rows (or columns) of the Young tableaux corresponding to the transpose of an n -dimensional p -order tensor cannot exceed n . This completes the counting work of transpose isomers.

The conclusion is: In an n -dimensional space, the number of independent transpositions of p -order tensors is equal to the number of standard Young tableaux pairs with p elements and columns less than or equal to n . Only when $n \geq p$, would all transposes be independent.

REFERENCES

- [1] Smith, G. F. Lectures on constitutive expressions. Banach Center Publications, 1985, 1.15: 645-678.
- [2] Schensted, Craige. Longest increasing and decreasing subsequences. Canadian Journal of mathematics 13 (1961): 179-191.
- [3] D. E. Knuth. Permutations, matrices, and generalized Young tableaux. Pacific Journal of Mathematics, 1970, 24:709-727